Optimal Public Debt with Life Cycle Motives*

William B. Peterman
Federal Reserve Board

Erick Sager
Bureau of Labor Statistics

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Abstract

This paper shows that accounting for life cycle behavior substantially affects optimal public debt in the presence of incomplete markets. In a calibrated model, we find that the life cycle changes optimal policy from *public debt* equal to 24% of output to *public savings* equal to 61% of output because it introduces two features that are observed in the data: (i) young individuals have little wealth and accumulate savings during their lifetimes, and (ii) average consumption and hours worked vary over individuals' lifetimes. Public debt affects welfare by crowding out productive capital and increasing the interest rate, which encourages more self-insurance against labor market risk through private saving. Without the life cycle, the welfare benefits of public debt are larger since individuals simply have more wealth on average. With the life cycle, the welfare benefit is smaller because even though public debt leads to more private savings, individuals must accumulate this savings over their lifetimes. Instead, public savings improves welfare by yielding a lower interest rate that encourages a flatter allocation of consumption and leisure over individuals' lifetimes. Additionally, the life cycle makes optimal policy far less sensitive to wealth inequality because wealth is now correlated not only with income, but also with age.

Keywords: Government Debt; Life Cycle; Heterogeneous Agents; Incomplete Markets

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1 Introduction

Motivated by the prevalence of government borrowing across advanced economies, previous work demonstrates that government debt can be optimal in a standard incomplete markets model with infinitely lived agents. For example, in their seminal work, Aiyagari and McGrattan (1998) find that a large quantity of public debt is optimal when such an economic environment is calibrated to the U.S. economy. Public debt is optimal because it crowds out the stock of productive capital and leads to a higher interest rate that encourages households to save more. As a result, households are better self-insured against idiosyncratic labor earnings risk and are, therefore, less likely to be liquidity constrained. Yet, while household savings behavior is central to public debt being optimal, previous work largely examines optimal policy in economies inhabited by infinitely lived agents. Such economic environments abstract from empirically relevant life cycle characteristics that can affect optimal debt policy as a result.

This paper characterizes the effect of a life cycle on optimal public debt and inspects the mechanisms through which a life cycle affects optimal policy. In order to determine the effect of the life cycle, we contrast optimal policy in two model economies: (i) the standard incomplete markets model with infinitely lived agents, and (ii) a life cycle model. We find that the optimal policies are strikingly different between the two models. In the infinitely lived agent model, it is optimal for the government to be a net borrower with public debt equal to 24 percent of output. In contrast, in the life cycle model, we find that it is optimal for the government to be a net saver, not a net borrower, with public savings equal to 61 percent of output.

Accounting for life cycle behavior leads to the optimality of public savings, not public debt, through two features that are also observed in the data: (i) young individuals have little wealth and accumulate savings during their lifetimes, and (ii) average consumption and hours worked vary over individuals' lifetimes. The first feature, that life cycle agents accumulate savings during their lifetimes, *dampens* the main welfare benefit of public debt. In particular, public debt can improve welfare by inducing a higher interest rate that encourages agents to save. Without a life cycle, agents will live in an economy that has more private savings on average, which improves agents' self-insurance against idiosyncratic labor earnings risk. With a life cycle, although public debt also increases the average amount of private savings, agents enter the economy with little or no wealth and must accumulate savings over their lifetimes. Thus, public debt has a smaller welfare benefit from self-insurance in the life cycle model because agents only experience improved self-insurance after they have accumulated savings.

The second feature, that average consumption and hours worked vary over the life cycle instead of remaining constant, leads public savings to be optimal. This is because public savings promotes a better allocation of agents' consumption and hours over their lifetimes. Life cycle agents tend to increase their consumption over most of their lifetimes because the return on savings under the public debt policy is high relative to their discount factors.[1] Holding total lifetime consumption fixed, a public debt or savings policy that results in agents allocating consumption more equally throughout their lifetimes maximizes expected lifetime utility. Therefore, public savings can improve welfare in the life cycle model by inducing a lower interest rate that reduces consumption growth over most of the lifetime and leads to a flatter consumption profile.

In contrast to these two features that lead to a divergence, income inequality reduces the difference between the two models' optimal policies. Underlying this reduction are three relationships: (i) a change in public debt moves the interest rate and wage in opposite directions, (ii) income inequality is due to inequality in both asset and labor income, and (iii) generally, asset income inequality increases with the interest rate while labor income inequality increases with the wage. Put together, these three relationships imply that optimal policy trades-off decreasing income inequality from one income source with increasing income inequality from the other. Comparing the two models, the infinitely lived agent model features a larger ratio of asset income inequality to labor income inequality compared to the life cycle model.[2] Accordingly, in the infinitely lived agent model, a reduction in public debt improves welfare because the lower return to savings leads to a decrease in asset income inequality. Conversely, in the life cycle model, a decrease in public savings improves welfare because the lower return to labor income reduces labor income inequality. Despite its countervailing impact, we find that the effect of inequality on optimal policy is quantitatively smaller than the effect from the improved lifetime allocation of consumption and hours in the life cycle model. Thus, public savings is optimal in the

[1] Life cycle agents' discount factors not only include the standard time preference but also account for mortality risk. In our baseline calibration of the life cycle model, mortality rates are sufficiently low relative to the interest rate to generate rising consumption over most of the lifetime. However, mortality risk increases after age 70, which generates falling consumption over the remainder of the lifetime. A similar mechanism governs the non-constant labor allocation over the life cycle.

[2] Asset income inequality is smaller in the life cycle model due to a finite lifetime. As agents live longer, there are more opportunities for agents to receive a long string of positive or negative labor productivity shocks, which generate greater dispersion in savings between lucky and unlucky agents. In the life cycle model, the finite lifetime means there are fewer opportunities for labor productivity shocks to propagate into the wealth distribution and create more variance in wealth.

life cycle model and public debt is optimal in the infinitely lived agent model.

Our results demonstrate that studying optimal policy in an infinitely lived agent model, which abstracts from the realism of a life cycle in order to render models more computationally tractable, is not without loss of generality. Not only is the optimal policy quite different when one ignores life cycle characteristics, but the welfare consequences of ignoring them are economically significant. In the life cycle model, we find that if a government is a net borrower (as is optimal in the infinitely lived agent model) instead of being a net saver (as is optimal in the life cycle model), then an average life cycle agent would be worse off by 0.6 percent of expected lifetime consumption.

This paper is related to an established literature that uses the standard incomplete market model with infinitely lived agents, originally developed in Bewley (1986), İmrohoroğlu (1989), Huggett (1993), Aiyagari (1994) and others, to study the optimal level of steady state government debt. Unlike this paper, previous work has mostly utilized infinitely lived agent models and finds that public debt is optimal. Aiyagari and McGrattan (1998) is the seminal contribution to the study of optimal debt in the standard incomplete market model and finds that public debt is optimal in an economy calibrated to resemble the U.S. Floden (2001) finds that increasing government debt can improve welfare if transfers are below optimal levels. Similarly, Dyrda and Pedroni (2016) find that it is optimal for the government to be a net borrower. However, they find that optimizing both taxes and debt at the same time leads to a smaller level of optimal debt than do previous studies. Relative to these papers, we focus on how optimal policy changes when one considers a life cycle model as opposed to an infinitely lived agent model, and find that including life cycle features has large effects on optimal policy.[3]

Using variants of incomplete market models, Röhrs and Winter (2017) and Vogel (2014) also find that it can be optimal for the government to be a net saver. In both papers, the government's desire for redistribution partially explains the optimality of public savings, as public savings leads to a lower interest rate and therefore redistributes welfare from wealth-rich agents to wealth-poor agents.[4] Similarly, this

[3]Using infinitely lived agent models, Desbonnet and Weitzenblum (2012), Açikgöz (2015), Dyrda and Pedroni (2016), Röhrs and Winter (2017) find quantitatively large welfare costs of transitioning between steady states after a change in public debt. Moreover, Heinemann and Wulff (2017) demonstrate that debt-financed government stimulus after an aggregate shock can be welfare improving. We do not consider these transitional costs and instead focus on steady state comparisons to more sharply highlight the effect of the life cycle on optimal debt policy.

[4]This motive to redistribute is enhanced in both of these papers since the models are calibrated to match the upper tail of the U.S. wealth distribution, which leads to a small mass of wealth-rich agents and a larger mass of wealth-poor agents.

paper finds that the redistribution motive affects optimal policy. However, we find that whether the effect moves optimal policy towards public savings or public debt depends on whether a life cycle is included.[5] However, we find that the existence of the accumulation phase and the government's desire to induce agents to more equally allocate their consumption over their lifetime are quantitatively dominant, leading to the optimality of public savings in the life cycle model and the optimality of public debt in the infinitely lived agent model.[6] Moreover, we find that optimal policy is much less sensitive to inequality in the life cycle model than in the infinitely lived agent model because wealth is correlated not only with income, but also with age when the life cycle is included.

This paper is also related to a strand of literature that examines the effects of life cycle features on optimal fiscal policy but generally focuses on taxation instead of government debt. For example, Garriga (2001), Erosa and Gervais (2002) and Conesa et al. (2009) show that life cycle characteristics create a motive for positive capital taxation, in contrast to the seminal work by Judd (1985) and Chamley (1986) that finds optimal capital taxation is zero in the long-run of a class of infinitely lived agent models.[7] With a life cycle, if age-dependent taxation is not feasible then a positive capital tax may be optimal since it can mimic an age-dependent tax on labor income. Instead of focusing on optimal taxation in a life cycle model, this paper quantifies the effects of life cycle features on optimal government debt.[8] We find that introducing life cycle features changes optimal policy from public debt to public savings because agents must accumulate savings at the beginning of their lifetimes, not because the government would like to mimic age-dependent policy.

Finally, this paper is related to Dávila, Hong, Krusell, and Ríos-Rull (2012), whose

[5]Specifically, we find that the ratio of asset income inequality relative to lifetime labor income inequality increases with the length of the lifetime (see Dávila et al. (2012) for discussion). Thus, in the infinitely lived agent model, there is a stronger desire for the government to reduce lifetime interest income inequality, which it can accomplish through public savings that lowers the interest rate. In contrast, in the life cycle model, there is more desire for the government to reduce lifetime labor income inequality, which it can accomplish by increasing public debt and thereby lowering the wage.

[6]In characterizing optimal public debt, this paper additionally abstracts from aggregate uncertainty (i.e., Barro (1979), Lucas and Stokey (1983), Aiyagari, Marcet, Sargent, and Seppälä (2002), Shin (2006)), political economy distortions (i.e., Alesina and Tabellini (1990), Battaglini and Coate (2008), and Song, Storesletten, and Zilibotti (2012)), and international capital flows (i.e., Azzimonti, de Francisco, and Quadrini (2014)).

[7]In addition, Aiyagari (1995) and İmrohoroğlu (1998) demonstrate that incomplete markets can overturn the zero capital tax result with uninsurable earnings shocks and sufficiently tight borrowing constraints.

[8]Instead of isolating the effects of life cycle features on optimal debt, Garriga (2001) allows the government to choose sequences for taxes (capital, labor and consumption) as well as government debt. In contrast, our paper explicitly measures how including life cycle features alters optimal debt policy while holding other fiscal instruments constant.

work defines constrained efficiency in a standard incomplete markets model with infinitely lived agents. Constrained efficient allocations account for the effect of individual behavior on market clearing prices, while satisfying individuals' constraints. The authors show that the price system in the standard incomplete market model does not efficiently allocate resources across agents and that welfare improving equilibrium prices could be attained if agents were to systematically deviate from individually optimal savings and consumption decisions. While this paper does not characterize constrained efficient allocations, this paper's Ramsey allocation improves welfare for similar reasons: since it understands the relationship between public debt and prices, the government can implement a welfare improving allocation that individual agents cannot attain through private markets. Because of this common mechanism, both of our papers find that a higher capital stock improves welfare. However, Dávila et al. (2012) obtains this result through matching top wealth inequality in an infinitely lived agent model, while our paper does so through adding life cycle features.[9]

The remainder of this paper is organized as follows. Section 2 illustrates the underlying mechanisms by which optimal government policy interacts with life cycle and infinitely lived agent model features. Section 3 describes the life cycle and infinitely lived agent model environments and defines equilibrium. Section 4 explains the calibration strategy, Section 5 presents quantitative results, and Section 6 performs robustness exercises. Section 7 concludes.

2 Illustration of the Mechanisms

In this section, we illustrate the mechanisms that lead the government to an optimal public debt or savings policy. We discuss why optimal government policy may differ between the life cycle and infinitely lived agent models.

2.1 Life Cycle Phases

In order to highlight how the life cycle may impact optimal debt policy, it will be useful to define three distinct phases of an agent's life cycle. Agents enter the economy with little or no wealth and begin the *accumulation phase*, which is characterized by the accumulation of wealth for precautionary motives and to finance post-retirement

[9] If we increase the amount of wealth inequality in the life cycle model using the same modeling strategy as Dávila et al. (2012), then we find wealth inequality leads to a larger quantity of optimal public savings than in the baseline life cycle model that does not explicitly match top wealth inequality.

consumption. While accumulating a stock of savings, agents may choose to work more and consume less.

Once wealth provides sufficient insurance against labor productivity shocks, agents have entered the *stationary phase*.[10] This phase is characterized by savings, hours and consumption that remain constant on average.[11]

Finally, agents enter the *deaccumulation phase* in old age. Late in the life cycle, agents deaccumulate assets because they no longer want to hold as much savings for precautionary reasons. As a result, the average level of savings decreases, average consumption may increase, and average hours worked may decrease.

In comparison, infinitely lived agents only experience the equivalent of a stationary phase. On average, infinitely lived agents' consumption, hours and savings allocations remain constant.

2.2 Welfare Channels and Life Cycle Features

We identify four main channels through which public debt policy affects welfare: the *direct effect*, the *insurance channel*, the *inequality channel* and the *age-allocation channel*. We heuristically characterize how these channels affect optimal policy, and how these channels' effects can differ in the life cycle and infinitely lived agent models.

Direct Effect: The *direct effect* is the partial equilibrium change in the productive capital stock, aggregate consumption, and aggregate output with respect to a change in public debt, when holding constant the aggregate labor supply and aggregate private savings. Mechanically, increased public debt crowds out (e.g., decreases) productive capital, thereby generating less output and decreasing aggregate consumption.[12] Generally, decreased aggregate consumption reduces welfare, which causes this mechanism to push optimal policy toward public savings. Absent any general equilibrium

[10]The stationary level of average savings is related to the "target savings level" in Carroll (1992, 1997). Given the primitives of the economy, an agent faces a tradeoff between consumption levels and consumption smoothing. The agent targets a level of savings that provides sufficient insurance while maximizing expected consumption.

[11]However, underlying constant averages for the cohort are individual agents who respond to shocks by choosing different allocations, thereby moving about various states within a non-degenerate distribution over savings, hours and consumption. If mortality is stochastic and the probability varies over the lifetime, then the cohort averages for savings, hours, and consumption will not be constant in this phase.

[12]By assuming that a representative firm operates a standard Cobb-Douglas production technology, aggregate output is a decreasing function of capital and labor inputs. Standard parameter assumptions ensure that steady state aggregate investment decreases by less than aggregate output decreases upon capital crowd out. Therefore, the resource constraint implies that aggregate consumption decreases.

effects, this mechanism should operate similarly in both the life cycle and infinitely lived agent economies.

Indirect Effects: While the direct effect is a partial equilibrium effect of policy on aggregate resources, the remaining channels affect welfare in general equilibrium, that is, by impacting market clearing prices. In particular, decreasing public savings or increasing public debt will crowd out productive capital and lead to an increase in the market clearing interest rate and a reduction in the market clearing wage rate.

An increase in the interest rate encourages agents to save. The higher level of savings improves welfare because agents are less likely to face binding liquidity constraints and are, therefore, better insured against labor earnings risk. We refer to this channel as the *insurance channel*.

The insurance channel's welfare benefit varies substantially between the life cycle and infinitely lived agent models. In the infinitely lived agent model, agents exist in a perpetual stationary phase. This implies that since public debt encourages agents to save, it leads to an equilibrium in which agents have more savings on average. Thus, increased public debt improves insurance for the average agent because he lives with more *ex ante* savings. In the life cycle model, in contrast, agents enter the economy with little or no wealth and immediately begin the accumulation phase.[13] While increased public debt may encourage agents to save more over their lifetime, agents need to accumulate this savings in the first place which reduces the benefit from public debt.[14]

The second indirect effect concerns how changes in the interest rate affect the allocation of consumption over the life cycle. The lower interest rate associated with public savings will lead agents to prefer using more resources for consumption today as opposed to saving resources for consumption at a later age. Thus, if consumption tends to increase over the lifetime, then the lower interest rate associated with public savings will induce a flatter consumption profile. Similarly, if consumption tends to

[13]If life cycle features were introduced in a dynastic model, instead of a life cycle model, where old agents bequeath wealth to agents entering the economy, then the accumulation phase may be more responsive to public policy. Consistent with Fuster, İmrohoroğlu, and İmrohoroğlu (2008), the optimal policy differences between a dynastic model and the infinitely lived agent model could be smaller than the difference between the optimal policies in the life cycle model and infinitely lived agent model since agents would receive some initial wealth through bequests.

[14]Savings accumulation mitigates the welfare benefit from the insurance for two reasons. First, life cycle agents only realize the insurance benefit from precautionary savings after accumulating that savings. Second, although the higher interest rate associated with public debt may encourage agents to accumulate a higher level of savings by the time they enter the stationary phase, agents will need to work more and consume less during the accumulation phase in order to reach a higher level of stationary savings.

decline over the lifetime, then public savings and the associated higher interest rate would lead to steeper fall in consumption. In a standard consumption-savings problem, abstracting from changes to the level of total lifetime consumption, public debt or public savings policy that results in agents allocating consumption more equally throughout their lifetimes maximizes expected lifetime utility.[15] We refer to this as the *age-allocation channel*. The age-allocation channel only exists in the life cycle model since there is no meaningful concept of age in the infinitely lived agent model.

The final indirect channel describes the welfare effect of income inequality arising from price changes. Income inequality is composed of both asset and labor income inequality, and the amount of asset income inequality increases with the interest rate while the amount of labor income inequality increases with the wage. Since changing public debt has opposite effects on the wage and interest rate, optimal policy trades-off decreasing income inequality from one income source with increasing income inequality from the other. Therefore, the optimal tradeoff depends on the relative amount of inequality that arises from each source of income. We refer to this channel as the *inequality channel*.

Since the relative inequality deriving from labor income and asset income varies across the two models, so too will the optimal policy tradeoff. As demonstrated in Dávila, Hong, Krusell, and Ríos-Rull (2012), inequality depends on agents' lifespan. As agents live longer, lifetime labor income inequality increases because there is a greater chance that agents receive a long string of either positive or negative labor productivity shocks. However, asset income inequality will also develop because agents reduce (increase) their wealth in response to a string of negative (positive) shocks. Generally, as each agent's lifespan increases, asset income inequality increases relative to labor income inequality. Accordingly, the ratio of asset income inequality to labor income inequality is larger in the infinitely lived agent model, and smaller in the life cycle model. Therefore, the inequality channel pushes optimal policy in the life cycle model toward more public debt (less public savings) and pushes optimal policy in the infinitely lived agent model toward more public savings (less public debt).

Overall, given these competing mechanisms, it is a quantitative issue whether public debt or public savings is optimal in either model. Put together, higher public debt lowers welfare through the direct effect and raises welfare through the insurance channel, while the age-allocation channel and inequality channel have ambiguous effects. Comparing the effects of the channels in the two models, the inequality channel's

[15]Similarly, changes in the interest rate can change how agents choose to allocate their leisure and labor over their lifetime and how a smooth allocation of labor/leisure will maximize utility. However, for simplicity, we choose to focus on describing the intuition with the allocation of consumption.

effect differs, pushing the life cycle model's optimal policy toward public debt and the infinitely lived agent model's optimal policy toward public savings. Likewise, the insurance channel is weaker in the life cycle model than in the infinitely lived agent model because the existence of the accumulation phase, while the infinitely lived agent model does not have an age-allocation effect. Thus, it is unclear whether introducing a life cycle will move optimal policy towards more public debt or towards public savings. In the next section, we turn to a quantitative model in order to determine the relative strength of these mechanisms.

3 Economic Environment

In this section, we present both the life cycle model and the infinitely lived agent model. Given that there are many common features across models, we will first focus on the life cycle model in detail before providing an overview of the infinitely lived agent model.

3.1 Life Cycle Model

3.1.1 Production

We assume there exists a large number of firms that sell a single consumption good in a perfectly competitive product market, purchase inputs from perfectly competitive factor markets, and each operate an identical constant returns to scale production technology, $Y = ZF(K, L)$. These assumptions on primitives admit a representative firm that chooses capital (K) and labor (L) inputs in order to maximize profits, given an interest rate r, a wage rate w, a level of total factor productivity Z, and capital depreciation rate $\delta \in (0, 1)$.

3.1.2 Consumers

Demographics: Let time be discrete and let each model period represent a year. Each period, the economy is inhabited by J overlapping generations of individuals. We index agents' age in the model by $j = 1, \ldots, J$, where $j = 1$ corresponds to age 21 in the data and J is an exogenously set maximum age (set to age 100 in the data). Before age J all living agents face mortality risk. Conditional on living to age j, agents have a probability ψ_j of living to age $j + 1$, with a terminal-age survival probability given by $\psi_J = 0$. Each period a new cohort is born and the size of each successive newly born cohort grows at a constant rate $g_n > 0$.

Agents who die before age J may hold savings when they die since mortality is uncertain. These savings are treated as accidental bequests and are equally divided across each living agent in the form of a lump-sum transfer, denoted Tr.

Preferences: Agents enjoy lifetime paths of consumption and labor, denoted $\{c_j, h_j\}_{j=1}^{J}$, according to the following preferences:

$$\mathbb{E}_1 \sum_{j=1}^{J} \beta^{j-1} \psi_j \left[u(c_j) - v(h_j, d_j) \right]$$

where β is the time discount factor, and $u(c)$ and $v(h)$ are instantaneous utility functions over consumption and labor hours, respectively, satisfying standard conditions. Furthermore, expectations are taken with respect to the stochastic processes governing labor productivity.

Finally, d_j denotes a retirement status. Agents choose their retirement age, which is denoted by J_{ret}, in the interval $j \in [\underline{J}_{ret}, \bar{J}_{ret}]$ and are forced to retire after age $\bar{J}_{ret}$. Therefore, let $d_j \equiv \mathbb{1}(j < J_{ret})$ be an indicator variable that equals one when an agent chooses to continue working and zero upon retirement. A retired agent cannot sell labor hours and the retirement decision is irreversible.

Labor Earnings: Agents are endowed with one unit of time per period, which they split between leisure and market labor. During each period of working life, an agent's labor earnings are $we_j h_j$, where w is the wage rate per efficiency unit of labor, e_j is the agent's idiosyncratic labor productivity drawn at age j, and h_j is the time the agent chooses to work at age j.

Following Kaplan (2012), we assume that labor productivity shocks can be decomposed into four sources:

$$\log(e_j) = \kappa + \theta_j + \nu_j + \epsilon_j$$

where (i) $\kappa \overset{iid}{\sim} \mathcal{N}(0, \sigma_\kappa^2)$ is an individual-specific fixed effect that is drawn once at birth and remains fixed, (ii) $\{\theta_j\}_{j=1}^{J}$ is an age-specific fixed effect that evolves in a predetermined manner, (iii) ν_j is a persistent shock that follows an autoregressive process given by $\nu_{j+1} = \rho \nu_j + \eta_{j+1}$ with $\eta \overset{iid}{\sim} \mathcal{N}(0, \sigma_\nu^2)$ and $\eta_1 = 0$, and (iv) $\epsilon_j \overset{iid}{\sim} \mathcal{N}(0, \sigma_\epsilon^2)$ is a transitory shock that is drawn each period.

For notational compactness, we denote the relevant state as a vector $\varepsilon_j = (\kappa, \theta_j, \nu_j, \epsilon_j)$ that contains each element necessary for computing contemporaneous labor earnings, $e_j \equiv e(\varepsilon_j)$, and forming expectations about future labor earnings. Denote the Markov process governing the process for ε by $\pi_j(\varepsilon_{j+1}|\varepsilon_j)$ for each ε_j, ε_{j+1} and for each $j = 1, \ldots, \bar{J}_{ret}$.

Assets: Agents have access to a single asset, a non-contingent one-period bond denoted a_j with a market determined rate of return of r. Agents may take on a net debt position, in which case they are subject to a borrowing constraint that requires their debt position be bounded below by $\underline{a} \in \mathbb{R}$. Agents are endowed with zero initial wealth, such that $a_1 = 0$ for each agent.

3.1.3 Government Policy

The government (i) consumes an exogenous amount of resources G, (ii) collects linear Social Security taxes τ_{ss} on all pre-tax labor income below an amount $\bar{m}$, (iii) distributes lump-sum Social Security payments b_{ss} to retired agents, (iv) distributes accidental bequests as lump-sum transfers Tr, and (v) collects income taxes from each individual.

Social Security: The model's Social Security system consists of taxes and payments. The social security payroll tax is given by τ_{ss} with a per-period cap denoted by $\bar{m}$. We assume that half of the social security contributions are paid by the employee and half by the employer. Therefore, the consumer pays a payroll tax given by: $(1/2)\ \tau_{ss} \min\{weh, \bar{m}\}$. Social security payments are computed using the averaged indexed monthly earnings (AIME) that summarizes an agents lifetime labor earnings. Following Huggett and Parra (2010) and Kitao (2014), the AIME is denoted by $\{m_j\}_{j=1}^{J}$, has an initial value $m_1 = 0$ and evolves as follows:

$$m_{j+1} = \left\{ \begin{array}{ll} \frac{1}{j}\left(\min\{we_jh_j, \bar{m}\} + (j-1)m_j\right) & \text{for } j \leq 35 \\ \max\left\{m_j, \frac{1}{j}\left(\min\{we_jh_j, \bar{m}\} + (j-1)m_j\right)\right\} & \text{for } j \in (35, J_{ret}) \\ m_j & \text{for } j \geq J_{ret} \end{array} \right\}$$

The AIME is a state variable for determining future benefits. Benefits consist of a base payment and an adjusted final payment. The base payment, denoted by $b^{ss}_{base}(m_{J_{ret}})$, is computed as a piecewise-linear function over the individual's average labor earnings at retirement $m_{J_{ret}}$:

$$b^{ss}_{base}(m_{J_{ret}}) = \left\{ \begin{array}{lll} \tau_{r1} m_{J_{ret}} & \text{for} & m_{J_{ret}} \in [0, b^{ss}_1) \\ \tau_{r1} b^{ss}_1 + \tau_{r2}(m_{J_{ret}} - b^{ss}_1) & \text{for} & m_{J_{ret}} \in [b^{ss}_1, b^{ss}_2) \\ \tau_{r1} b^{ss}_1 + \tau_{r2} b^{ss}_2 + \tau_{r3}(m_{J_{ret}} - b^{ss}_1 - b^{ss}_2) & \text{for} & m_{J_{ret}} \in [b^{ss}_2, b^{ss}_3) \\ \tau_{r1} b^{ss}_1 + \tau_{r2} b^{ss}_2 + \tau_{r3} b^{ss}_3 & \text{for} & m_{J_{ret}} \geq b^{ss}_3 \end{array} \right\}$$

Lastly, the final payment requires an adjustment that penalizes early retirement and credits delayed retirement. The adjustment is given by:

$$b_{ss}(m_{J_{ret}}) = \left\{ \begin{array}{lll} (1 - D_1(J_{nra} - J_{ret}))b^{ss}_{base}(m_{J_{ret}}) & \text{for} & \underline{J}_{ret} \leq J_{ret} < J_{nra} \\ (1 + D_2(J_{ret} - J_{nra}))b^{ss}_{base}(m_{J_{ret}}) & \text{for} & J_{nra} \leq J_{ret} \leq \bar{J}_{ret} \end{array} \right\}$$

where $D_i(\cdot)$ are functions governing the benefits penalty or credit, $\underline{J}_{ret}$ is the earliest age agents can retire, J_{nra} is the "normal retirement age," and $\bar{J}_{ret}$ is the latest age an agent can retire.

Income Taxation: Taxable income is defined as labor income and capital income net of social security contributions from an employer:

$$y(h, a, \varepsilon, d) \equiv \left\{ \begin{array}{ll} we(\varepsilon)h + r(a + Tr) - \frac{\tau_{ss}}{2} \min\{we(\varepsilon)h, \bar{m}\} & \text{if } \ d = 1 \\ r(a + Tr) & \text{if } \ d = 0 \end{array} \right.$$

The government taxes each individual's taxable income according to an increasing and concave function, $\Upsilon(y(h, a, e, d))$.

Public Savings and Budget Balance: Each period, the government has a debt balance B and saves or borrows (denoted B') at the market interest rate r. If the government borrows, then $B' < 0$ and the government repays rB' next period. If the government saves, then $B' > 0$ and the government collects asset income rB' next period. The resulting government budget constraint is:

$$G + B' - B = rB + R \tag{1}$$

where R is aggregate revenues from income taxation and G is an unproductive level of government expenditures.[16] The model's Social Security system is self-financing

[16]Two recent papers, Röhrs and Winter (2017) and Chaterjee, Gibson, and Rioja (2016) have relaxed the

and therefore does not appear in the governmental budget constraint.

3.1.4 Consumer's Problem

The agent's state variables consist of asset holdings a, labor productivity shocks $\varepsilon \equiv (\kappa, \theta, \nu, \epsilon)$, Social Security contribution (AIME) variable m, and retirement status d_{-1}. The age-j agent's recursive problem prior to retirement is:

$$V_j(a, \varepsilon, m, 1) = \max_{c, a', h, d} \left[u(c) - v(h, d)\right] + \beta\psi_j \sum_{\varepsilon'} \pi_j(\varepsilon'|\varepsilon) V_{j+1}(a', \varepsilon', m', d) \tag{2}$$

$$\begin{aligned} \text{s.t.} \quad c + a' &\leq we(\varepsilon)h + (1+r)(a + Tr) - \tfrac{\tau_{ss}}{2}\min\{we(\varepsilon)h, \bar{m}\} - \Upsilon(y(h, a, \varepsilon, d)) \\ a' &\geq \underline{a} \end{aligned}$$

An agent retires when they choose $d = 0$ between ages $\underline{J}_{ret}$ and $\bar{J}_{ret}$ or face mandatory retirement after age $\bar{J}_{ret}$. The retired agent's recursive problem is:

$$V_j(a, 1, m, 0) = \max_{c, a', h} \left[u(c) - v(h, 0)\right] + \beta\psi_j V_{j+1}(a', 1, m, 0) \tag{3}$$

$$\begin{aligned} \text{s.t.} \quad c + a' &\leq (1+r)(a + Tr) + b_{ss}(m) - \Upsilon\left(r(a + Tr)\right) \\ a' &\geq \underline{a} \end{aligned}$$

3.1.5 Recursive Competitive Equilibrium

Agents are heterogeneous with respect to their age $j \in \mathbf{J} \equiv \{1, \ldots, J\}$, wealth $a \in \mathbf{A}$, labor productivity $\varepsilon \in \mathbf{E}$, average lifetime earnings $m \in \mathbf{X}$, and retirement status $d \in \mathbf{D} \equiv \{0, 1\}$. Let $\mathbf{S} \equiv \mathbf{A} \times \mathbf{E} \times \mathbf{X} \times \mathbf{D}$ be the state space and $\mathcal{B}(\mathbf{S})$ be the Borel σ-algebra on $\mathbf{S}$. Let $\mathbf{M}$ be the set of probability measures on $(\mathbf{S}, \mathcal{B}(\mathbf{S}))$. Then $(\mathbf{S}, \mathcal{B}(\mathbf{S}), \lambda_j)$ is a probability space in which $\lambda_j(S) \in \mathbf{M}$ is a probability measure defined on subsets of the state space, $S \in \mathcal{B}(\mathbf{S})$, that describes the distribution of individual states across age-j agents. Denote the fraction of the population that is age $j \in \mathbf{J}$ by μ_j. For each set $S \in \mathcal{B}(\mathbf{S})$, $\mu_j\lambda_j(S)$ is the fraction of age $j \in \mathbf{J}$ and type $S \in \mathbf{S}$ agents in the economy. We can now define a recursive competitive equilibrium of the economy.

Definition (Equilibrium): Given a government policy $(G, B, B', R, \tau_{ss}, b_{ss})$, a *stationary recursive competitive equilibrium* is (i) an allocation for consumers described by policy

standard Ramsey assumption that government expenditures are unproductive. Both papers show that public savings is optimal with productive government expenditures, intuitively because there is an additional benefit to aggregate output.

functions $\{c_j, a'_j, h_j, d_j\}_{j=1}^J$ and consumer value function $\{V_j\}_{j=1}^J$, (ii) an allocation for the representative firm (K, L), (iii) prices (w, r), (iv) accidental bequests Tr, and (v) distributions over agents' state vector at each age $\{\lambda_j\}_{j=1}^J$ that satisfy:

(a) Given prices, policies and accidental bequests, $V_j(a, \varepsilon, m, d_{-1})$ solves the Bellman equation (2) with associated policy functions $c_j(a, \varepsilon, m, d_{-1})$, $a'_j(a, \varepsilon, m, d_{-1})$, $h_j(a, \varepsilon, m, d_{-1})$ and $d_j(a, \varepsilon, m, d_{-1})$.

(b) Given prices (w, r), the representative firm's allocation minimizes cost, $r = ZF_K(K, L) - \delta$ and $w = ZF_L(K, L)$.

(c) Accidental bequests, Tr, from agents who die at the end of this period are distributed equally across next period's living agents:

$$(1 + g_n)Tr = \sum_{j=1}^{J} (1 - \psi_j)\mu_j \int a'_j(a, \varepsilon, m, d_{-1}) \mathbf{d}\lambda_j(a, \varepsilon, m, d_{-1}). \tag{4}$$

(d) Government policies satisfy budget balance in equation (1), where aggregate income tax revenue is given by:

$$R \equiv \sum_{j=1}^{J} \mu_j \int \Upsilon\left(y\big(h_j(a, \varepsilon, m, d_{-1}), a, \varepsilon, d_j(a, \varepsilon, m, d_{-1})\big)\right) \mathbf{d}\lambda_j(a, \varepsilon, m, d_{-1}). \tag{5}$$

(e) Social security is self-financing:

$$\sum_{j=1}^{J} \mu_j \int d_j(a, \varepsilon, m, d_{-1}) \tau_{ss} \min\{we(\varepsilon)h_j(a, \varepsilon, m, d_{-1}), \bar{m}\} \mathbf{d}\lambda_j(a, \varepsilon, m, d_{-1})$$

$$= \sum_{j=1}^{J} \mu_j \int (1 - d_j(a, \varepsilon, m, d_{-1})) b_{ss}(m) \mathbf{d}\lambda_j(a, \varepsilon, m, d_{-1}). \tag{6}$$

(f) Given policies and allocations, prices clear asset and labor markets:

$$K - B = \sum_{j=1}^{J} \mu_j \int a \ \mathbf{d}\lambda_j(a, \varepsilon, m, d_{-1}) \tag{7}$$

$$L = \sum_{j=1}^{J} \mu_j \int d_j(a, \varepsilon, m, d_{-1}) e(\varepsilon) h_j(a, \varepsilon, m, d_{-1}) \ \mathbf{d}\lambda_j(a, \varepsilon, m, d_{-1}) \tag{8}$$

and the allocation satisfies the resource constraint (guaranteed by Walras' Law):

$$\sum_{j=1}^{J} \mu_j \int c_j(a,\varepsilon,m,d_{-1})\mathbf{d}\lambda_j(a,\varepsilon,m,d_{-1}) + G + K' = ZF(K,L) + (1-\delta)K. \quad (9)$$

(g) Given consumer policy functions, distributions across age j agents $\{\lambda_j\}_{j=1}^{J}$ are given recursively from the law of motion $T_j^* : \mathbf{M} \to \mathbf{M}$ for all $j \in \mathbf{J}$ such that T_j^* is given by:

$$\lambda_{j+1}(\mathcal{A} \times \mathcal{E} \times \mathcal{X} \times \mathcal{D}) = \sum_{d_{-1}\in\{0,1\}} \int_{A\times E\times X} Q_j\left((a,\varepsilon,m,d_{-1}), \mathcal{A} \times \mathcal{E} \times \mathcal{X} \times \mathcal{D}\right) \mathbf{d}\lambda_j$$

where $\mathcal{S} \equiv \mathcal{A} \times \mathcal{E} \times \mathcal{X} \times \mathcal{D} \subset \mathbf{S}$, and $Q_j : \mathbf{S} \times \mathcal{B}(\mathbf{S}) \to [0,1]$ is a transition function on $(\mathbf{S}, \mathcal{B}(\mathbf{S}))$ that gives the probability that an age-j agent with current state $\mathbf{s} \equiv (a,\varepsilon,m,d_{-1})$ transits to the set $\mathcal{S} \subset \mathbf{S}$ at age $j+1$. The transition function is given by:

$$Q_j\left((a,\varepsilon,m,d_{-1}),\mathcal{S}\right) = \left\{ \begin{array}{ll} \psi_j \cdot \pi_j(\mathcal{E}|\varepsilon)^{d_{-1}} & \text{if } a_j'(\mathbf{s}) \in \mathcal{A},\ m_j'(\mathbf{s}) \in \mathcal{X},\ d_j(\mathbf{s}) \in \mathcal{D} \\ 0 & \text{otherwise} \end{array} \right\}$$

where agents that continue working and transition to set $\mathcal{E}$ choose $d_j(\mathbf{s}) = 1$, while agents that transition from working life to retirement choose $d_j(\mathbf{s}) = 0$. For $j = 1$, the distribution λ_j reflects the invariant distribution $\pi_{ss}(\varepsilon)$ of initial labor productivity over $\varepsilon = (\kappa, \theta_1, 0, \epsilon_1)$.

(h) Aggregate capital, governmental debt, prices and the distribution over consumers are stationary, such that $K' = K$, $B' = B$, $w' = w$, $r' = r$, and $\lambda_j' = \lambda_j$ for all $j \in \mathbf{J}$.

3.2 Infinitely Lived Agent Model

The infinitely lived agent model differs from the life cycle model in three ways. First, agents in the infinitely lived agent model lifetimes are infinite ($J \to \infty$) and agents have no mortality risk. Second, labor productivity no longer has an age-dependent component ($\theta_j = \bar{\theta}$ for all $j \geq 1$). Lastly, there is no retirement ($J_{ret} \to \infty$ such that $d_j = 1$ for all $j \geq 1$) and there is no Social Security program ($\tau_{ss} = 0$ and $b_{ss}(m) = 0$ for all x).

Accordingly, we study a stationary recursive competitive equilibrium in which the initial endowment of wealth and labor productivity shocks no longer affects individual decisions and the distribution over wealth and labor productivity is time invariant.

Definition (Equilibrium): Given a government policy (G, B, B', R), a *stationary recursive competitive equilibrium* is (i) an allocation for consumers described by policy functions (c, a', h) and consumer value function V, (ii) an allocation for the representative firm (K, L), (iii) prices (w, r), and (v) a distribution over agents' state vector λ that satisfy:

(a) Given prices and policies, $V(a, \varepsilon)$ solves the following Bellman equation:

$$V(a,\varepsilon) = \max_{c,a',h} \; \left[u(c) - v(h)\right] + \beta \sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon) V(a', \varepsilon') \tag{10}$$

$$\begin{aligned} \text{s.t.} \quad c + a' &\leq we(\varepsilon)h + (1+r)a - \Upsilon(y(h,a,\varepsilon)) \\ a' &\geq \underline{a} \end{aligned}$$

with associated policy functions $c(a, \varepsilon)$, $a'(a, \varepsilon)$ and $h(a, \varepsilon)$.

(b) Given prices (w, r), the representative firm's allocation minimizes cost.

(c) Government policies satisfy budget balance in equation (1), where aggregate income tax revenue is given by:

$$R \equiv \int \Upsilon\big(y(h(a,\varepsilon), a, \varepsilon)\big) \, \mathbf{d}\lambda(a,\varepsilon) \tag{11}$$

(d) Given policies and allocations, prices clear asset and labor markets:

$$K - B = \int a \; \mathbf{d}\lambda(a,\varepsilon) \tag{12}$$

$$L = \int e(\varepsilon) h(a,\varepsilon) \; \mathbf{d}\lambda(a,\varepsilon) \tag{13}$$

and the allocation satisfies the resource constraint (guaranteed by Walras' Law):

$$\int c(a,\varepsilon) \mathbf{d}\lambda(a,\varepsilon) + G + K' = ZF(K,L) + (1-\delta)K \tag{14}$$

(e) Given consumer policy functions, the distribution over wealth and productivity shocks is given recursively from the law of motion $T^* : \mathbf{M} \to \mathbf{M}$ such that T^* is given by:

$$\lambda'(\mathcal{A} \times \mathcal{E}) = \int_{A \times E} Q_j\left((a,\varepsilon), \mathcal{A} \times \mathcal{E}\right) \mathbf{d}\lambda$$

where $\mathcal{S} \equiv \mathcal{A} \times \mathcal{E} \subset \mathbf{S}$, and $Q : \mathbf{S} \times \mathcal{B}(\mathbf{S}) \rightarrow [0,1]$ is a transition function on $(\mathbf{S}, \mathcal{B}(\mathbf{S}))$ that gives the probability that an agent with current state $\mathbf{s} \equiv (a, \varepsilon)$ transits to the set $\mathcal{S} \subset \mathbf{S}$ in the next period. The transition function is given by:

$$Q\left((a,\varepsilon), \mathcal{S}\right) = \begin{Bmatrix} \pi(\mathcal{E}|\varepsilon) & \text{if } a'(\mathbf{s}) \in \mathcal{A}, \\ 0 & \text{otherwise} \end{Bmatrix}$$

(f) Aggregate capital, governmental debt, prices and the distribution over consumers are stationary, such that $K' = K$, $B' = B$, $w' = w$, $r' = r$, and $\lambda' = \lambda$.

3.3 Balanced Growth Path

Following Aiyagari and McGrattan (1998), we will further assume that total factor productivity, Z, grows over time at rate $g_z > 0$. In both the life cycle model and infinitely lived agent model, we will study a balanced growth path equilibrium in which all aggregate variables grow at the same rate as output. Denote the growth rate of output as g_y. Refer to Appendix A for a formal construction of the balanced growth path for this set of economies.

4 Calibration

In this section we calibrate the life cycle model and then discuss the parameter values that are different in the infinitely lived agent model. Overall, one subset of parameters are assigned values without needing to solve the model. These parameters are generally the same in both models. The other subset of parameters are estimated using a simulated method of moments procedure that minimizes the distance between model generated moments and empirical ones. We allow these parameters to vary across the models while matching the same moments in the two models. Table 1 summarizes the target and value for each parameter.

Demographics: Agents enter the economy at age 20 (or model age $j = 1$) and exogenously die at age 100 (or model age $J = 81$). We set the conditional survival probabilities $\{\psi_j\}_{j=1}^{J}$ according to Bell and Miller (2002) and impose $\psi_J = 0$. We set the population growth rate to $g_n = 0.011$ to match annual population growth in the US.

Production: We assume that the aggregate production function is Cobb-Douglas of the form $F(K, L) = K^{\alpha} L^{1-\alpha}$ where $\alpha = 0.36$ is the income share accruing to capital. Total factor productivity is normalized to one, $Z = 1$. The depreciation rate is set to $\delta = 0.0833$ which allows the model to match the empirically observed investment-to-output ratio.

Preferences: The utility function is separable in the utility over consumption and disutility over labor (including retirement):

$$u(c) - v(h, d) = \frac{c^{1-\sigma}}{1-\sigma} - \left(\chi_1 \frac{h^{1+\frac{1}{\gamma}}}{1+\frac{1}{\gamma}} + d\chi_2 \right) .$$

Utility over consumption is a CRRA specification with a coefficient of relative risk aversion $\sigma = 2$, which is consistent with Conesa et al. (2009) and Aiyagari and McGrattan (1998). Disutility over labor exhibits a constant intensive margin Frisch elasticity. We choose $\gamma = 0.5$ such that the Frisch elasticity consistent with the majority of the related literature as well as the estimates in Kaplan (2012).

We calibrate the labor disutility parameter χ_1 so that the cross sectional average of hours is one third of the time endowment. Finally, χ_2 is a fixed utility cost of not being retired. The fixed cost generates an extensive margin decision through a non-convexity in the utility function. We choose χ_2 to match the empirical observation that seventy percent of the population has retired by the normal retirement age.

Labor Productivity Process: We take the labor productivity process from the estimates in Kaplan (2012).[17] The deterministic labor productivity profile, $\{\theta_j\}_{j=1}^{\bar{J}_{ret}}$, is (i) smoothed by fitting a quadratic function in age, (ii) normalized such that the value equals unity when an agent enters the economy, and (iii) extended to cover ages 20 through 70, which we define as the last period in which agents are able to participate

[17]For details on estimation of this process, see Appendix E in Kaplan (2012). A well known problem with a log-normal income process is that the model generated wealth inequality does not match that in the data, primarily due to missing the fat upper tail of the distribution. However, Röhrs and Winter (2017) demonstrate that when the income process in an infinitely lived agent model is altered to match the both the labor earnings and wealth distributions (quintiles and gini coefficients), the change in optimal policy is relatively small, with approximately 30 percentage points due to changing the income process and the remaining 110 percentage points due to changing borrowing limits, taxes and invariant parameters (such as risk aversion, the Frisch elasticity, output growth rate and depreciation).

in the labor activities ($\bar{J}_{ret}$).[18] The permanent, persistent, and transitory idiosyncratic shocks to individual's productivity are normally distributed with zero mean. The remaining parameters are also set in accordance with the Kaplan's (2012) estimates: $\rho = 0.958$, $\sigma_\kappa^2 = 0.065$, $\sigma_\nu^2 = 0.017$ and $\sigma_\epsilon^2 = 0.081$.

Government: Consistent with Aiyagari and McGrattan (1998) we set government debt equal to two-thirds of output. We set government consumption equal to 15.5 percent of output consistent. This ratio corresponds to the average of government expenditures to GDP from 1998 through 2007.[19]

Income Taxation: The income tax function and parameter values are from Gouveia and Strauss (1994). The functional form is:

$$\Upsilon(y) = \tau_0 \left(y - \left(y^{-\tau_1} + \tau_2 \right)^{-\frac{1}{\tau_1}} \right)$$

The authors find that $\tau_0 = 0.258$ and $\tau_1 = 0.768$ closely match the U.S. tax data. When calibrating the model we set τ_2 such that the government budget constraint is satisfied.

Social Security: We set the normal retirement age to 66. Consistent with the minimum and maximum retirement ages in the U.S. Social Security system, we set the interval in which agents can retire to the ages 62 and 70. The early retirement penalty and delayed retirement credits are set in accordance with the Social Security program and define the functions $D_1(\cdot)$ and $D_2(\cdot)$. In particular, if agents retire up to three years before the normal retirement age, then agents' benefits are reduced by 6.7 percent for each year they retire early. If they choose to retire four or five years before the normal retirement age, then benefits are reduced by an additional 5 percent for these years. If agents choose to delay retirement past normal retirement age, then their benefits are increased by 8 percent for each year they delay. The marginal replacement rates in the progressive Social Security payment schedule ($\tau_{r1}, \tau_{r2}, \tau_{r3}$) are also set at their actual respective values of 0.9, 0.32 and 0.15. The bend points where the marginal replacement rates change (b_1^{ss}, b_2^{ss}, b_3^{ss}) and the maximum earnings ($\bar{m}$) are set equal to the actual multiples of mean earnings used in the U.S. Social Security system so that b_1^{ss}, b_2^{ss} and $b_3^{ss} = \bar{m}$ occur at 0.21, 1.29 and 2.42 times average earnings in the economy. We set the payroll tax rate, τ_{ss} such that the program's budget is bal-

[18]The estimates in Kaplan (2012) are available for ages 25-65.

[19]We exclude government expenditures on Social Security since they are explicitly included in our model.

Table 1: Calibration Targets and Parameters for Baseline Life Cycle Economy.

Description	Parameter	Value	Target or Source
Demographics			
Maximum Age	J	81 (100)	By Assumption
Min/Max Retirement Age	$\underline{J}_{ret}, \bar{J}_{ret}$	43, 51 (62, 70)	Social Security Program
Population Growth	g_n	1.1%	Conesa et al (2009)
Survival Rate	$\{\psi_j\}_{j=1}^{J}$	—	Bell and Miller (2002)
Preferences and Borrowing			
Coefficient of RRA	σ	2.0	Kaplan (2012)
Frisch Elasticity	γ	0.5	Kaplan (2012)
Coefficient of Labor Disutility	χ_1	56.2	Avg. Hours Worked = 1/3
Fixed Utility Cost of Labor	χ_2	1.048	70% retire by NRA
Discount Factor	β	1.012	Capital/Output = 2.7
Borrowing Limit	$\underline{a}$	0	By Assumption
Technology			
Capital Share	α	0.36	NIPA
Capital Depreciation Rate	δ	0.0833	Investment/Output = 0.255
Productivity Level	Z	1	Normalization
Output Growth	g_y	1.85%	NIPA
Labor Productivity			
Persistent Shock, autocorrelation	ρ	0.958	Kaplan (2012)
Persistent Shock, variance	σ_ν^2	0.017	Kaplan (2012)
Permanent Shock, variance	σ_κ^2	0.065	Kaplan (2012)
Transitory Shock, variance	σ_ϵ^2	0.081	Kaplan (2012)
Mean Earnings, Age Profile	$\{\theta\}_{j=1}^{\bar{J}_{ret}}$	—	Kaplan (2012)
Government Budget			
Government Consumption	G/Y	0.155	NIPA Average 1998-2007
Government Savings	B/Y	-0.667	NIPA Average 1998-2007
Marginal Income Tax	τ_0	0.258	Gouveia and Strauss (1994)
Income Tax Progressivity	τ_1	0.786	Gouveia and Strauss (1994)
Income Tax Progressivity	τ_2	4.648	Balanced Budget
Social Security			
Payroll Tax	τ_{ss}	0.103	Social Security Program
SS Replacement Rates	$\{\tau_{ri}\}_{i=1}^{3}$	See Text	Social Security Program
SS Replacement Bend Points	$\{b_i^{ss}\}_{i=1}^{3}$	See Text	Social Security Program
SS Early Retirement Penalty	$\{D_i\}_{i=1}^{2}$	See Text	Social Security Program

anced. In our baseline model the payroll tax rate is 10.3 percent, roughly equivalent with the statutory rate.[20]

Infinitely Lived Agent Model: The infinitely lived agent model does not have an age-dependent wage profile. For comparability across models, we replace the age-dependent wage profile with the population-weighted average of θ_j's, such that $\bar{\theta} =$

[20] Although the payroll tax rate in the U.S. economy is slightly higher than our calibrated value, the OASDI program includes additional features outside of the retirement benefits.

$\sum_{j=1}^{J_{ret}}(\mu_j / \sum_{j=1}^{J_{ret}} \mu_j)\theta_j \approx 1.86.$[21] In the absence of a retirement decision, we set $\chi_2 = 0$. Lastly, we recalibrate the parameters (β, χ_1) to the same targets as in the life cycle model and choose τ_2 to balance the government's budget, obtaining $\beta = 0.967$, $\chi_1 = 34.7$ and $\tau_2 = 3.133$.

5 Quantitative Effects of the Life Cycle on Optimal Policy

This section computes optimal public debt policy in the life cycle and infinitely lived agent models and quantifies the contribution of the life cycle to policy differences. We quantify the effect of each life cycle feature through the construction of a series of counterfactual models that systematically removes life cycle features until recovering the infinitely lived agent model. Therefore the counterfactual models isolate the response of optimal policy to each life cycle model component. We conclude this section by computing the welfare effect from implementing a public savings policy instead of a public debt policy. Moreover, we decompose this welfare effect in order to further highlight the implications of life cycle features for optimal policy.

5.1 Optimal Public Policy

In both the life cycle and infinitely lived agent models, the government is a benevolent Ramsey planner that fully commits to fiscal policy. The planner maximizes social welfare by choosing a budget feasible level of public savings ($B > 0$) or public debt ($B < 0$) subject to allocations being a stationary recursive competitive equilibrium. We consider an ex-ante Utilitarian social welfare criterion that evaluates the expected utility of an agent in the steady state economy.[22]

For the life cycle model, the Ramsey planner chooses public savings to maximize

[21] When calibrating the stochastic process for idiosyncratic productivity shocks, we use the same process in the both the life cycle and infinitely lived agent models. Using the same underlying process will imply that cross-sectional wealth inequality will be different across the two models. One reason is that the life cycle model will have additional cross-sectional inequality due to the humped shaped savings profiles, which induces the accumulation, stationary, and deaccumulation phases. We view these difference in inequality as a fundamental difference between the two models and, therefore, choose not to specially alter the infinitely lived agent model to match a higher level of cross-sectional inequality.

[22] Our analysis focuses on welfare across steady states. This analysis omits the transitional costs between steady states, which can be large. See Domeij and Heathcote (2004), Fehr and Kindermann (2015) and Dyrda and Pedroni (2016) for a discussion of these transitional costs.

the expected lifetime utility of newborn agents as follows,

$$S_J(V_1, \lambda_1) \equiv \max_B \left\{ \int V_1(a, \varepsilon, m, d_{-1}; B) \, \mathbf{d}\lambda_1(a, \varepsilon, m, d_{-1}; B) \quad \text{s.t.} \quad (1),\ (6) \right\}$$

where the value function $V_1(\cdot; B)$, distribution function $\lambda_1(\cdot; B)$, and policy functions embedded in equations (1) and (6) are determined in competitive equilibrium and depend on the planner's choice of public savings. Furthermore, $B' = B$ in steady state. Since the distribution of taxable income and tax revenues depend on public savings, we adjust the income tax parameter τ_0 and the payroll tax rate τ_{ss} to ensure that the government budget is balanced and Social Security is self-financing.[23]

For the infinitely lived agent model, the Ramsey planner chooses public savings to maximize the expected utility of infinitely lived agents as follows,

$$S_\infty(V, \lambda) \equiv \max_B \left\{ \int V(a, \varepsilon; B) \, \mathbf{d}\lambda(a, \varepsilon; B) \quad \text{s.t.} \quad G = rB + R(\tau_0, B) \right\}$$

The welfare maximization problem is nearly identical to that of the life cycle model's, except that the value function and distribution function do not depend on age and there is no Social Security program, so that equation (6) does not characterize the feasible set.

We find that the two models generate starkly different optimal policies, which are reported in Table 2. In the infinitely lived agent model, the government is a net borrower with optimal public debt equal to 24 percent of output.[24] On the other hand, in the life cycle model, the government's optimal policy is public savings equal to 61 percent of output. Thus, including life cycle features causes optimal policy to switch from public debt to savings, with approximately an 85 percentage point swing in optimal policy.

We quantify the welfare gain from implementing optimal policy. In particular, we compute *consumption equivalent variation (CEV)* – the percent of lifetime consumption that a life cycle model agent would be willing to pay ex ante – from inhabiting an economy with an optimal public savings policy of 61 percent of output instead of an economy with the infinitely lived agent model's optimal public debt policy of 24 percent of output. We find that the 85 percentage point difference in optimal poli-

[23]We choose to use τ_0 to balance the government budget instead of the other income taxation parameters (τ_1, τ_2) so that the average income tax rate is used to clear the budget, as opposed to changing in the progressivity of the income tax policy. The average tax rate is the closest analogue to the flat tax that Aiyagari and McGrattan (1998) use to balance the government's budget in their model.

[24]This is generally consistent with Aiyagari and McGrattan's (1998) optimal policy. The differences in optimal policy are due to this paper's assumption of a different stochastic process governing labor productivity, a different utility function, non-linear income taxation and different parameter values.

Table 2: Aggregates and Prices Across Models

	Life Cycle		Infinitely Lived	
	Baseline	**Optimal**	**Baseline**	**Optimal**
Public Savings/Output	**-0.67**	**0.61**	**-0.67**	**-0.24**
Consumption	0.53	0.54	0.66	0.66
Hours	0.33	0.33	0.33	0.34
Output	0.92	1.00	1.16	1.18
Productive Capital	2.50	3.03	3.13	3.23
Labor	0.53	0.54	0.66	0.67
Private Savings	3.11	2.41	3.90	3.51
Public Savings	-0.62	0.61	-0.77	-0.28
Interest Rate	5.0%	3.6%	5.0%	4.8%
Wage	1.12	1.19	1.12	1.13

cies corresponds to a welfare gain of 0.64 percent of expected lifetime consumption. The welfare gain is economically significant, demonstrating that ignoring life cycle features when determining optimal debt policy can have nontrivial welfare effects.

5.2 The Effect of Life Cycle Features on Optimal Policy

The 85 percentage point difference in optimal policies is due to the three main differences between the life cycle and infinitely lived agent models: (i) agents in the life cycle model experience all three life cycle phases, including an accumulation phase, while agents in the infinitely lived agent model experience a perpetual stationary phase, (ii) other age-dependent features in the life cycle model, such as mortality risk, an age-dependent wage profile, retirement and Social Security, do not exist in the infinitely lived agent model, and (iii) agents' lifespans differ between the two models.

In order to characterize the individual effects of these three differences on optimal policy we compute optimal policy in two counterfactual economies that systematically remove life cycle features (see Table 3).[25] The first counterfactual model is the "No Age-Dependent Features" economy, which is similar to the life cycle model but excludes all age-dependent features (e.g., mortality risk, age-dependent wage profile, retirement and Social Security system) while maintaining the maximum lifespan of $J = 81$ periods. Comparing the life cycle model with the "No Age-Dependent Fea-

[25]In order to make quantitative comparisons across models, each counterfactual model's parameters are recalibrated to match all relevant the targets described in Section 4.

Table 3: Optimal Public Savings-to-Output (Percent)

	Counterfactual Models		
Life Cycle	**No Age Features**	**Infinitely Lived with Accumulation**	**Infinitely Lived**
61%	201%	248%	-24%

tures" economy primarily isolates the effect of increasing expected working lifetime, due to removing retirement and mortality, on optimal policy. We find that optimal public savings increases from 61 percent of output in the life cycle model to 200 percent of output in the "No Age-Depend Features" economy.

The second counterfactual economy is the "Infinitely Lived with Accumulation" economy, which also excludes the age-dependent model features but additionally extends each agent's lifespan to $J = 1000$ periods. Extending lifespan from 81 periods to 1000 periods yields an approximation to an infinite lifespan, and comparing the "No Age-Dependent Features" and "Infinitely Lived with Accumulation" counterfactual economies isolates the direct effect of further increasing agents' lifespans. This effect additionally increases optimal public savings from 201 to 248 percent of output.

In the "Infinitely Lived with Accumulation" economy, agents enter the economy with no wealth. However, an agent's lifespan is long enough that this economy approximates the ex ante expected lifetime utility of an infinitely lived agent, yet still includes an accumulation phase.[26] Thus, comparing the "Infinitely Lived with Accumulation" counterfactual economy with the infinitely lived agent model isolates the effect of the accumulation phase on optimal policy, which changes optimal policy from public savings to public debt equal to 24 percent of output.

The optimal policies across these four models yield two notable results. First, removing life cycle features creates counterfactual models (e.g., the "No Age-Dependent Features" and "Infinitely Lived with Accumulation" counterfactual economies) that become increasingly similar to the infinitely lived agent model, yet optimal policy diverges from that in the infinitely lived agent model. In particular, removing life cycle features generates more optimal public savings relative to the life cycle model, instead of generating public debt (or less public savings) as is optimal in the infinitely lived agent model. Second, by comparing optimal policies from the "Infinitely Lived with Accumulation" economy and the infinitely lived agent model, we observe that

[26] We select $J = 1000$ for the lifespan in the "Infinitely Lived with Accumulation" economy because it is sufficiently large to ensure that a newborn's expected present value of the flow of utility from the end of the lifetime is essentially zero.

Table 4: Effect of Lifespan on Inequality (Coefficient of Variation)

	Life Cycle	**No Age Features**	**Infinitely Lived w/ Accumulation**	**Infinitely Lived**
Asset Income Inequality	0.63	0.61	0.79	1.05
Labor Income Inequality	0.31	0.28	0.28	0.35
$\frac{\textbf{Asset Income Inequality}}{\textbf{Labor Income Inequality}}$	2.03	2.15	2.89	2.98

removing the accumulation phase accounts for a 272 basis point change in optimal policy, between 248% public savings to 24% public debt. These results highlight two competing effects on optimal policy from life cycle features: (i) the differential effect of the inequality channel across models, and (ii) the effect of life cycle phases, which are absent from infinitely lived agent model. We next discuss these competing effects in turn.

First, the inequality channel has a differential effect on optimal policy in the two models because the amount of labor income inequality relative to asset income inequality generally depends on agents' lifespans. As agents live and work longer, asset income inequality tends to rise relative to labor income inequality because there is more time for labor productivity shocks to propagate into the wealth distribution and enlarge the difference in wealth between lucky and unlucky agents. Note that relative to the life cycle model, agents in the "No Age-Dependent" and "Infinitely Lived with Accumulation" counterfactual models work for a longer length of time (e.g., due to removing mortality and retirement, or mechanically extending lifespan). Table 4 confirms that with the longer working lifetime, asset income inequality relative to labor income inequality is larger in both counterfactual economies than it is in the standard life cycle model under the baseline public debt policy of 67% of output.

Since government policy affects the returns from labor and capital in opposite directions, optimal policy trades off reducing income inequality from the source for which the factor price decreases with increasing income inequality from the source for which the factor price increases. Thus, the counterfactual models have higher levels of optimal public savings than does the life cycle model, because asset income inequality rises relative to labor income inequality as we remove life cycle features and extend agents' expected working lifetimes. The percent change in lifetime total income inequality in Table 5 confirms that, in fact, adopting optimal policy reduces total income inequality relative to the baseline public debt policy of 67% of output.[27]

[27]In the baseline life cycle model, in which asset income inequality is relatively smaller compared

Table 5: Lifetime Total Income Inequality (Coefficient of Variation)

	Life Cycle	No Age Features	Infinitely Lived w/ Accumulation	Infinitely Lived
Baseline Policy	0.36	0.33	0.32	0.50
Optimal Policy	0.35	0.31	0.28	0.48
Percent Change	-0.7%	-6.3%	-12.7%	-3.4%

Thus, the inequality channel causes the optimal level of public savings to increase after eliminating life cycle features but while retaining the accumulation phase.

The second competing effect, due to the existence of the accumulation phase in the life cycle model, is the primary model feature that leads to the optimality of public savings instead of public debt in the life cycle model. Comparing optimal policies in the "Infinitely Lived with Accumulation" with the infinitely lived agent models isolates the effect of accumulation phase, which leads to a 272 percentage point difference in optimal policy (as reported in Table 3). To further isolate this effect, we conduct a computational experiment in which we compute the optimal policy of the "Infinitely Lived with Accumulation" counterfactual model according to an alternative social welfare function that only incorporates the expected present value of utility after a given age $j^* > 1$, and ignores the flow of utility from ages 1 to $j^* - 1$. Thus, as j^* increases, the social welfare function ignores more of the accumulation phase.[28]

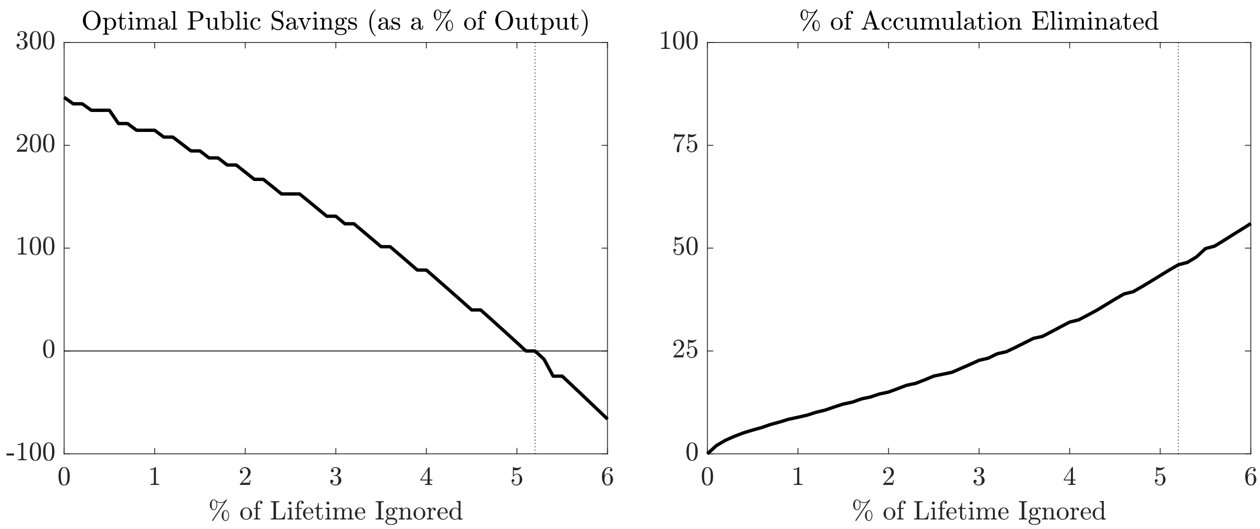
The computational experiment demonstrates that as the accumulation phase matters less for social welfare, optimal policy tends toward more public debt, as shown in Figure 1. The left panel in Figure 1 plots the optimal policy of the "Infinitely Lived with Accumulation" counterfactual model under the alternative welfare criterion, while the right panel plots the percentage of the accumulation phase that is ignored when computing optimal policy. On the x-axis, both graphs vary the percent of lifetime that the threshold age j^* represents. We observe that optimal policy mono-

to labor income inequality, we find that adopting public savings reduces total income inequality. However, we find in Section 5.3 that in terms of welfare, this change increases inequality.

[28]Specifically, government policy maximizes agents' expected lifetime utility as of age j^*, subject to allocations being determined in competitive equilibrium, as follows:

$$\tilde{S}(V_{j^*}, \lambda_{j^*}) \equiv \max_B \left\{ \int V_{j^*}(a, \varepsilon; B) \, \mathbf{d}\lambda_{j^*}(a, \varepsilon; B) \quad \text{s.t.} \quad G = rB + R(\tau_0, B) \right\}.$$

Figure 1: Optimal Policy and Eliminating Accumulation



Notes: The left panel graphs the optimal public savings to output ratio (y-axis) associated with ignoring a given percent of early life utility flows (x-axis). The percent of "Lifetime Ignored" is measured as $100 \cdot (j^*/J)$, using the given value of j^* and $J = 1000$. The right panel graphs the percent of accumulation that is eliminated under the optimal policy associated with ignoring a given percent of early life utility flows. The percent of eliminated wealth accumulation is defined as the average private savings of j^*-age agents (given a particular optimal public savings policy) relative to the peak average savings (under the baseline public debt policy) and converted to a percent. The vertical dashed line demarcates the percent of early lifetime utility ignored at which optimal policy switches from public savings to debt.

tonically decreases from public savings of 248 percent of GDP when all of the lifetime is considered, to an optimal public debt policy when the social welfare function ignores at least 5.2 percent of agents' early lifetime, or approximately 45 percent of the accumulation phase.[29]

There are two reasons why eliminating the accumulation phase leads to the optimality of public debt. First, the accumulation phase mitigates the welfare benefit from the insurance channel, since agents only experience improved self-insurance after they have accumulated the additional savings associated with a higher interest rate. In contrast, by eliminating the accumulation phase, agents experience a perpetual stationary phase in which increased public debt simply increases their ex ante

[29]Similarly, we find public savings is optimal in the infinitely lived agent model when the government only considers the welfare agents that closely resemble life cycle agent entering the accumulation phase. Specifically, if the government only places Pareto weight on the set of borrowing constrained agents with the median persistent component of the labor productivity shock, then optimal public savings equals 300 percent of output. This result reinforces the notion that wealth accumulation reduces the welfare benefit from public debt.

Table 6: Welfare Decompositions

	Life Cycle (% Change)	**Infinitely Lived (% Change)**
Overall CEV	0.64	-0.12
Level (Δ_{level})	-0.23	-0.17
Age (Δ_{age})	1.08	0
Distribution (Δ_{distr})	-0.20	0.05

Notes: The life cycle and infinitely lived agent model welfare decompositions compare allocations under a 24% public debt policy with a 61% public savings policy.

wealth. Therefore, eliminating the accumulation phase strengthens the welfare benefit from public debt through the insurance channel. Second, when agents only experience the stationary phase, average consumption and hours are constant instead of varying systematically with age. As a result, there is no longer a welfare benefit from public savings leading agents to more equally allocate consumption and hours over their lifetimes. Therefore, as the accumulation phase is eliminated, it is optimal to reduce public savings and eventually adopt a public debt policy.

To summarize, we find that extending agents' working lifetime increases the amount of asset income inequality relative to labor income inequality, thereby pushing optimal policy toward public savings. However, eliminating the accumulation phase enhances the welfare benefit from public debt by strengthening the insurance channel and weakening the age-allocation channel. Overall, we find that the effects of the accumulation phase dominate the effects of other life cycle model features on optimal policy, ultimately resulting in the optimality of public savings in the life cycle model and the optimality of public debt in the infinitely lived agent model.

5.3 Welfare Decomposition

In order to quantify the effects of the main channels (see Section 2.2) that lead public debt to be optimal in the infinitely lived agent model and public savings to be optimal in the life cycle model, we examine the welfare implications from adopting public savings instead of public debt. Specifically, we quantify the welfare effects from the 85 percentage point change, from the optimal public debt policy in the infinitely lived agent model to the optimal public savings policy in the life cycle model.

The welfare effects from changing public policy reflect the change in aggregate resources available to agents and the allocation of those resources across agents and across their lifetimes. Thus, we decompose the consumption equivalent variation (denoted Δ_{CEV}) into a *level effect* (Δ_{level}), an *age effect* (Δ_{age}) and a *distribution effect* (Δ_{distr}).[30] The decomposition is defined as

$$(1 + \Delta_{CEV}) = (1 + \Delta_{level}) \cdot (1 + \Delta_{age}) \cdot (1 + \Delta_{distr}).$$

The overall CEV, Δ_{CEV}, is explicitly defined in equation (B5), while Δ_{level}, Δ_{age} and Δ_{distr} are explicitly defined in equation (B1) and equation (B6) through equation (B11) of Appendix B.

The level effect captures the welfare change for a fictitious "representative agent," absent any idiosyncratic or life cycle variation in consumption or hours. The age effect measures agents' change in welfare as a result of changing age-specific average levels of consumption and hours, net of changes in aggregate consumption and hours. Accordingly, the age effect captures the welfare effect of a change in the slope of the average consumption and hours age-profiles. Note that the age effect does not exist in the infinitely lived agent model and therefore infinitely lived agents attain zero welfare change through age effects. Lastly, the distribution effect measures the remaining change in welfare that results from a change in the distribution of consumption and hours across agents.

The welfare decomposition presented in Table 6 demonstrates that the age effect, which is closely tied to the age-allocation channel, is crucial for explaining why public savings increases welfare in the life cycle model but not in the infinitely lived agent model. In particular, the 0.64 percent welfare improvement from implementing public savings in the life cycle model is due to a 1.08 percent increase from the age effect that is partially offset by a 0.23 percent decrease from the level effect and a 0.20 percent decrease from the distribution effect. In contrast, the small 0.12 percent CEV loss in the infinitely lived agent model can be attributed to a larger negative level effect (0.17) than a positive distribution effect (0.05). There is no effect from the age-allocation channel in the infinitely lived agent model.

The life cycle model's positive age effect from adopting public savings is closely tied to the age-allocation channel and, accordingly, indicates an improved allocation of consumption and hours across ages. Agents possess standard concave utility func-

[30] Floden (2001) decomposes the CEV into a level effect, an insurance effect, a redistribution effect and an hours effect. Relative to Floden's (2001) decomposition, (i) we combine the insurance and redistribution effects to form the "distribution effect", (ii) we add an age effect, which only exists in the life cycle model, and (iii) because an extensive margin retirement decision is unique to the life cycle model, we incorporate both intensive and extensive margins into the welfare decomposition of the life cycle model's hours allocation. Appendix B formally derives the decomposition.

tions and prefer flat lifetime consumption and hours allocations. As shown in Figure 2, consumption tends to increase over a majority of the lifetime because agents choose to use more of their available resources for savings and delay consumption to later in their lifetimes. Similarly, labor hours decline over an average agent's working lifetime as agents choose to delay consumption of leisure. When deciding whether to consume today or save for tomorrow's consumption, agents must receive a sufficiently large interest rate to compensate for time preference, β, and mortality risk, ψ_j.[31] Thus, over a majority of the lifetime, the interest rate is sufficiently large to generate an upward sloping consumption profile.[32] However, the lower interest rate associated with pubic savings diminishes the returns to savings, thereby inducing agents to consume more while young. This change leads to a more equal allocation of consumption and hours over the lifetime, which improves welfare as seen in Figure 2. This channel does not exist in the infinitely lived agent model.

Adopting public savings leads to a welfare loss from the level channel. One reason that public savings reduces welfare through the level channel is that the lower interest rate associated with public savings means that agents hold less savings and thus are more likely to experience binding liquidity constraints. Constrained agents tend to experience a lower level of welfare since they tend to consume less and work more.

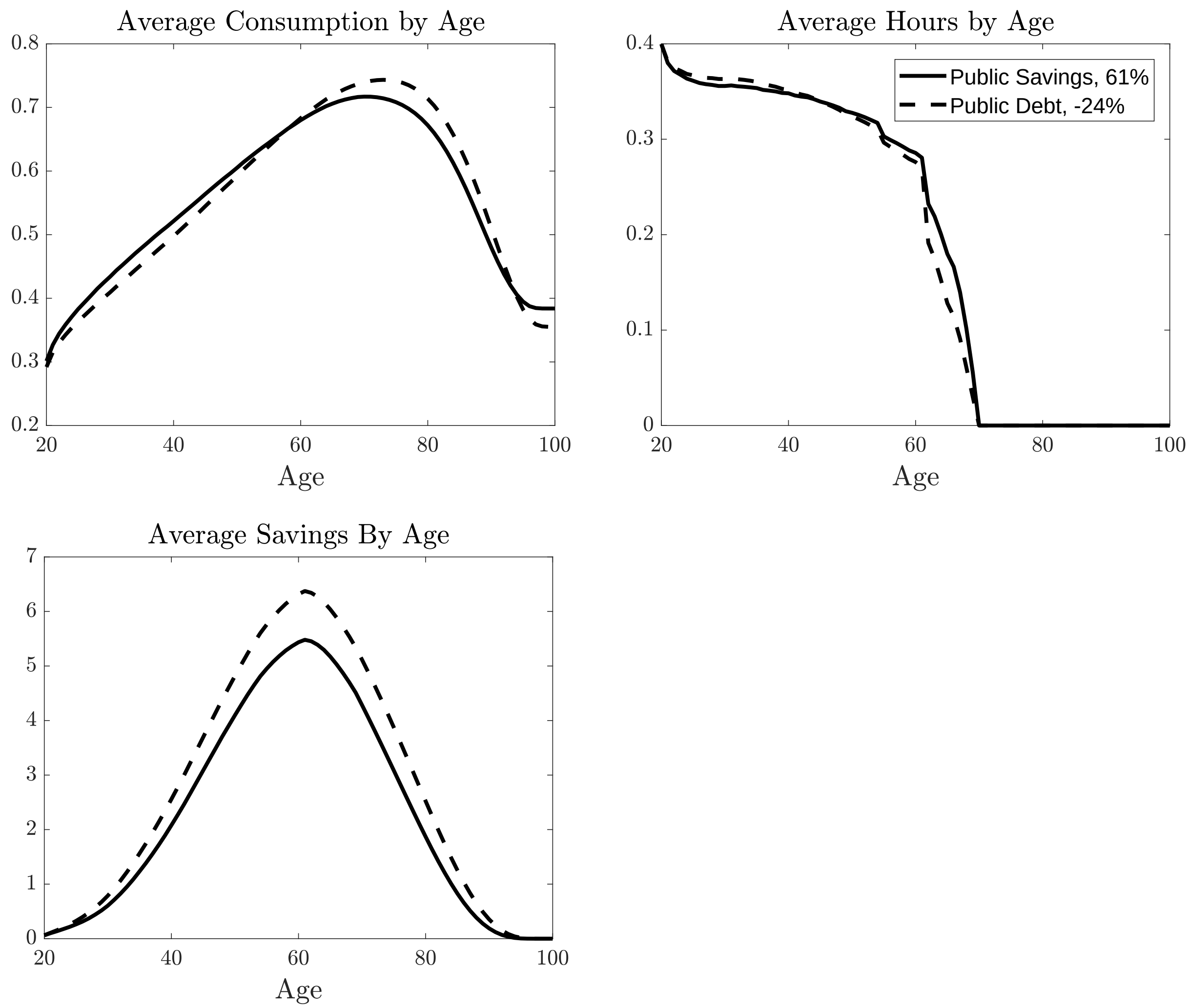
Finally, the distribution effect from adopting a public savings policy partially offsets the level and age effects in each model, leading to a welfare reduction in the life cycle model and a welfare improvement in the infinitely lived agent model. The difference in the distribution effect across models corresponds to the inequality channel. The higher wage and lower interest rate from public savings have different effects on inequality in the life cycle and infinitely lived agent models. As discussed in Section 2.2, a longer working lifetime in the infinitely lived agent model leads to more asset income inequality relative to labor earnings inequality. Thus, a higher wage and lower interest rate can reduce existing total income inequality. In the life cycle model, the opposite holds true; since asset income inequality relative to labor income inequality is smaller, a lower interest rate and higher wage exacerbate lifetime total income inequality.[33]

[31] Additionally, liquidity constraints can lead agents to consume less early in their lifetime. This effect is accentuated in the counterfactual infinitely lived with accumulation phase model in Section 5.2.

[32] The downward slopping labor supply profile is also affected by the age-dependent wage profile.

[33] In contrast to the age and distribution effect, the level effect from adopting public savings is similar in both models. In particular, there is a welfare increase from the consumption level effect and a welfare decrease from the hours level effect. Public savings leads to more productive capital so both output and consumption increase. However, the larger stock of productive capital leads to a higher wage which encourages more labor. Overall, the disutility from more labor dominates the increase in utility from more consumption because the lower interest rate associated with public savings reduces

Figure 2: Life Cycle Model: Consumption, Savings and Hours Profiles



Notes: Solid lines are cross-sectional averages for consumption, savings, and hours by age in the life cycle economy under its optimal public savings policy of 61%. The dashed lines are cross-sectional averages by age for the optimal debt policy from the infinitely lived agent economy of 24%.

To summarize, the welfare effects highlight the competing mechanisms that lead to different optimal policy across the life cycle and infinitely lived agent models. In the life cycle model, public savings encourages agents to more equally allocate their consumption and labor over their lifetime. However, the inequality channel decreases the difference in optimal policies across models, since adopting public savings increases wealth inequality in the life cycle model but reduces it in the infinitely lived agent model. On net, we find the quantitative magnitude of the benefits from public

the incentives for agents to save so they are more likely to face binding liquidity constraints (i.e. a reduction in the benefit from the insurance channel).

savings dominate in the life cycle model.

6 Wealth Inequality

While Section 5 quantifies the main channels that determine optimal debt policy, those results abstract from features that shape the bottom of the wealth distribution. This section reexamines optimal debt policy in the life cycle and infinitely lived agent models after introducing model ingredients that change the number of low wealth agents and the resources available to them. Specifically, this section shows how optimal policy responds to (i) allowing agents to borrow and (ii) varying the mass of agents with little or no wealth by altering the labor productivity process. We demonstrate that these features affect optimal policy quite differently in the life cycle and infinitely lived agent models.

6.1 Liquidity Constraints

In the benchmark model, we assumed that agents faced a no-borrowing constraint. In this section, we examine whether optimal policy is sensitive to allowing agents to borrow since borrowing may change the strength of the insurance channel and the age-allocation channel. To do this, we compute optimal policy in the life cycle and infinitely lived agent models when agents can borrow up to an exogenously set limit of 30 percent of each economy's aggregate private savings.[34] We find that allowing agents to borrow increases the difference between the optimal policies in the life cycle and infinitely lived agent models. While the infinitely lived agent model's optimal level of public debt remains approximately the same (28 percent of output), the life cycle model's optimal level of public savings is now twice as large (109 percent of output).

Allowing private borrowing leads to a large increase in optimal public savings in the life cycle model since it enables agents to more easily intertemporally smooth their consumption. In particular, young agents expect a sharp increase in their labor productivity during the beginning of their working lifetime due to the age-specific component ($\{\theta_j\}_{j=1}^{\bar{J}_{ret}}$). Since they enter the economy with low labor productivity and little to no wealth, agents would like to borrow against future income in order to flatten consumption and hours profiles over their lifetimes. When faced with a lower

[34] We calculate the limit as 30 percent of aggregate private savings under the baseline calibration. We choose to use the rest of the calibration parameters from the benchmark model since allowing borrowing has a very small quantitative impact on aggregate variables in both models.

interest rate associated with greater public savings, agents would like to borrow even more. Therefore, in order to not further exacerbate liquidity constraints, optimal public savings is limited to 61% of output when agents cannot borrow. However, when borrowing is allowed, agents are better able to intertemporally substitute consumption, which mitigates the negative effect of a lower interest rate on liquidity constraints. As a result, the optimal level of public savings doubles.

While life cycle agents' incentives to borrow derive from their increasing average labor productivity profile, infinitely lived agents experience a constant average labor productivity profile, $\theta_j = \bar{\theta}$ for all j. As a result, we find a minimal effect on optimal policy from allowing borrowing for these infinitely lived agents. Therefore, the main mechanism by which government debt improves welfare in the infinitely lived agent model is robust to changes in borrowing limits. However, in the case of the life cycle model, allowing borrowing strengthens the dominant mechanism and provides greater scope for public savings to improve welfare.

6.2 Labor Productivity Process

In this section, we match the fraction of agents with little or no wealth in the life cycle and infinitely lived agent models to what we observe in the Survey of Consumer Finances. Relative to the benchmark model, the wealth distribution will be more skewed (e.g., there will be a smaller fraction of agents holding a majority of the wealth and a larger fraction of the population that possesses little or no wealth). Increasing the fraction of low-wealth agents may alter optimal policy because the Ramsey planner will be more willing to implement a policy that benefits agents with little lifetime wealth, even at the expense of high-wealth agents.

Our quantitative implementation follows Castañeda, Díaz-Giménez, and Ríos-Rull (2003), and more recently Kindermann and Krueger (2014), by augmenting the standard log-normally distributed persistent labor productivity process (ν) in Section 4 with the addition of an extremely high labor productivity state.[35] We refer to this additional high labor productivity state as a *superstar shock*. We parameterize the superstar shock to make it unlikely but highly persistent. Accordingly, in the life cycle model, we set the probability of receiving the superstar shock to 1% and the

[35] Preference heterogeneity is an alternate way to introduce a skewed wealth. However, there are two downsides to using preference heterogeneity. First, it is unclear what discount rate should be used to measure social welfare. Second, in a model similar to ours that excludes altruism, Hendricks (2007) demonstrates that matching the wealth distribution requires including a large mass of both patient and impatient agents with a considerably larger gap in patience between these groups than is consistent with empirical estimates.

per-period persistence to 90%.[36] In the infinitely lived agent model, we set these two probabilities to match the duration and hazard rate of the superstar state that are implied by the life cycle model. We obtain a probability of receiving a superstar shock that is just over 1% and a probability of a superstar remaining a superstar in the next period equal to 86%. For both the life cycle and infinitely lived agent models, we choose the value of the superstar shock so that the bottom 60% of the population holds 5.4% of total wealth or, equivalently, the top 40% of the population holds 94.6% of total wealth (see Krueger, Mitman, and Perri (2016)). Since adding the superstar shock has large effects on the models, we calibrate all other model parameters to match the same targets used in the benchmark model (see Section 4).

When superstar shocks are included, we find that a large amount of public savings is optimal in both the infinitely lived and life cycle models. The life cycle model's optimal public savings equals 95% of output, which is a 34 percentage point change from the benchmark optimal policy of public savings equaling 61% of output. The infinitely lived agent model's optimal public saving policy equals 86% of output, which constitutes an even larger change from the benchmark optimal policy of public debt equaling 24% of output. Furthermore, Table 7 shows that there is a large welfare gain in both the life cycle and infinitely lived agent models from implementing the optimal public savings policies, relative to an economy with public debt equaling two-thirds of output.

Examining Table 7, although public savings is optimal in both the life cycle and infinitely lived agent models, the main welfare effect that leads to the optimality of public savings is different between models. In the life cycle model, the age effect still contributes the most to the overall welfare gain from implementing the optimal public savings policy. In the infinitely lived agent model, the strongly positive distribution effect leads public savings to be optimal.

Although different welfare effects dominate in each model, the distribution effect is now welfare improving in both the life cycle and infinitely lived agent models with superstar shocks. This stands in contrast to the benchmark life cycle model that excludes superstar shocks, in which the distribution effect reduces welfare (see Table 6).[37] Generally, superstars derive more of their total income from interest on savings than non-superstars and, therefore, a lower interest rate induced by public

[36] We assume that upon exiting the superstar state, agents transition to the median persistent labor productivity state. We further assume that no life cycle agent enters the economy as a superstar.

[37] Note that the CEV calculations in Table 7 and Table 6 compare different changes in policy. While this makes a comparison of magnitudes inappropriate, we can compare signs and shares since both CEV calculations compare a change from public debt to public savings policy.

Table 7: Welfare Decompositions with Superstar Shocks

	Life Cycle (% Change)	**Infinitely Lived (% Change)**
Overall CEV	3.49	1.18
Level (Δ_{level})	-1.16	-1.16
Age (Δ_{age})	3.95	0
Distribution (Δ_{distr})	0.72	2.36

Notes: The welfare decompositions compare allocations under a 67% public debt policy with the life cycle model and infinitely lived agent model optimal public savings policy of 95% and 86%, respectively.

savings redistributes from superstars to non-superstars.[38] On net, this redistribution from superstars to non-superstars raises welfare for two reasons. First, superstars have a very low marginal utility of consumption relative to non-superstars so redistributing between these types of agents leads to an increase in ex ante welfare.[39] Second, because receiving the superstar shock is a low probability event, there are far more agents who do not receive the superstar shock than there are agents who do. Therefore, redistribution benefits a significantly larger share of agents than it hurts.

Yet, while the addition of superstar shocks generates a positive welfare effect from redistribution in both models, the welfare effect is much stronger in the infinitely lived agent model than in the life cycle model. This is because agents may possess a *large* stock of wealth for different reasons in the two models. In the infinitely lived agent model, a large stock of wealth indicates that agents have recently experienced a superstar shock. Thus, public savings is effective at increasing the welfare of non-superstars relative to superstars through redistribution. In contrast, life cycle model agents can have a large stock of wealth either because they have recently experienced a superstar shock or because they are middle aged (i.e., have accumulated a lifetime of savings in preparation for retirement). As a result, public savings is less effective at isolating superstars in the life cycle model, which dampens the welfare gain from redistribution. Because public savings redistributes from superstars to non-superstars

[38]Superstars derive more income from savings because they tend to save a very high fraction of their income in order to maintain an elevated level of consumption even after reverting to non-superstars.

[39]In the life cycle model, young agents have little wealth and the low probability of being a superstars implies that there are few young superstars. Therefore, redistribution from high-wealth to low-wealth agents also intertemporally redistributes resources from agents' middle to early lifetime. As such, this intertemporal redistribution would be captured in the age effect in the welfare decomposition.

more effectively in the infinitely lived agent model than in the life cycle model, including superstar shocks generates a much larger change in optimal policy in the infinitely lived agent model (from 24% public debt to 86% public savings) than in the life cycle model (from 61% public savings to 95% public savings).

In both models adding superstar shocks means that optimal policy will decrease inequality through public savings. However, the potential welfare gain from reducing inequality due to public savings is overstated in the infinitely lived agent model. In particular, the infinitely lived agent model overstates welfare gains because it lacks the dampening effect from age-dependence that breaks the correlation between possessing high wealth and having received superstar shocks. Therefore, we find that the infinitely lived agent model is very sensitive to increased inequality, while the life cycle model's more realistic assumptions on age-dependence mean that the life cycle model is far less responsive to increased inequality through superstar shocks. In this respect, although optimal policy might be similar, the benefits from redistribution are overstated in the infinitely lived agent model while the benefits from enhanced intertemporal smoothing are underestimated.

7 Conclusion

This paper characterizes the effect of a life cycle on optimal public debt and evaluates the mechanisms by which a life cycle affects optimal policy. We find that the optimal policies are strikingly different between life cycle and infinitely lived agent models. We find that it is optimal for the government to be a *net saver* with savings equal to 61% of output when life cycle features are included. In contrast, it is optimal for the government to be a *net debtor* with debt equal to 24% of output when these life cycle features are excluded.

Furthermore, there are economically significant welfare consequences from not accounting for life cycle features when determining the optimal policy. We find that if a government implemented the infinitely lived agent model's optimal 24% debt-to-output policy in the life cycle model, then life cycle agents would be worse off by more than 0.6% of expected lifetime consumption.

We show that two salient empirical regularities in the life cycle model lead to public savings, not public debt, being optimal: (i) young individuals have little wealth and accumulate savings during their lifetimes, and (ii) instead of being constant, average consumption and hours worked vary over individuals' lifetimes. In the infinitely lived agent model, higher public debt implies that an average agent lives with more savings and is, therefore, better insured against labor earnings risk. In the life cycle model, in contrast, agents enter the economy with little or no wealth and must accumulating savings. While a higher level of public debt might encourage life cycle agents to hold more savings during their lifetimes, the fact that agents must accumulate this savings stock mitigates the welfare benefits from public debt. Moreover, the lower interest rate associated with public savings improves welfare in the life cycle model since it leads agents to more equally allocate their consumption across their lifetime. In contrast, this channel does not exist in the infinitely lived agent model since, from an ex ante perspective, expected consumption is flat over time.

We show that the particular progression of individual savings, consumption, and labor throughout the lifetime is the predominant reason for the drastically different optimal policies in the life cycle and infinitely lived agent models. In the infinitely lived agent model, higher public debt implies that an average agent begins each period of time with more savings and is, therefore, better insured against labor earnings risk. In the life cycle model, in contrast, agents enter the economy with little or no wealth and must accumulating savings. While a higher level of public debt might encourage life cycle agents to hold more savings during their lifetime, the fact that agents must accumulate this savings stock mitigates the welfare benefits from public debt. Moreover, the lower interest rate associated with public savings improves welfare in the life cycle model since it leads agents to more equally allocate their consumption across their lifetime. In contrast, this channel does not exist in the infinitely lived agent model since, from an ex ante perspective, expected consumption is flat over time.

When using quantitative models to answer economic questions, economists constantly face a trade-off between tractability and realism. Our results demonstrate that when examining the welfare consequences of public debt, it is not without loss of generality to utilize the more tractable infinitely lived agent model instead of a life cycle model.

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Appendix

A Construction of the Balanced Growth Path

This appendix provides a formal construction of the Balanced Growth Path for the set of economies described in Section 3. We construct the Balanced Growth Path in multiple parts. First we construct the Balanced Growth Path using aggregates from the models. Then, we construct the Balanced Growth Path using individual agents' allocations. The last two sections develop the Balanced Growth Path for any features unique to the infinitely lived agent or life cycle models.

A.1 Aggregate Conditions

Balanced Growth Path: A Balanced Growth Path (BGP) is a sequence

$$\{C_t, A_t, Y_t, K_t, L_t, B_t, G_t\}_{t=0}^{\infty}$$

such that (i) for all $t = 0, 1, \dots$ $C_t, A_t, Y_t, K_t, B_t, G_t$ grow at a constant rate g,

$$\frac{Y_{t+1}}{Y_t} = \frac{C_{t+1}}{C_t} = \frac{A_{t+1}}{A_t} = \frac{K_{t+1}}{K_t} = \frac{B_{t+1}}{B_t} = \frac{G_{t+1}}{G_t} = 1 + g$$

(ii) per capita variables all grow at the same constant rate g_w:

$$\frac{Y_{t+1}/N_{t+1}}{Y_t/N_t} = \frac{C_{t+1}/N_{t+1}}{C_t/N_t} = \frac{A_{t+1}/N_{t+1}}{A_t/N_t} = \frac{K_{t+1}/N_{t+1}}{K_t/N_t} = \frac{B_{t+1}/N_{t+1}}{B_t/N_t} = \frac{G_{t+1}/N_{t+1}}{G_t/N_t} = 1 + g_w$$

and (iii) effective labor per capita is constant:

$$\frac{L_{t+1}}{N_{t+1}} = \frac{L_t}{N_t} = \frac{L_0}{N_0}$$

Denote time 0 variables without a time subscript, for example $L \equiv L_0$.

Growth Rates: Let all growth derive from TFP $g_z > 0$ and population $g_n > 0$ growth. Then on a balanced growth path we assume:

$$Z_t = (1 + g_z)^t Z$$

$$N_t = (1 + g_n)^t N$$

where Z and N are steady state values. Then, from part (iii) of the definition, growth in labor is:

$$\frac{L_{t+1}}{L_t} = \frac{L_{t+1}/N_{t+1}}{L_t/((1+g_n)N_t)} = 1 + g_n$$

In steady state $Y = ZK^{\alpha}L^{1-\alpha}$. Let output growth be given by $g > 0$. Therefore the production function gives:

$$Y_t = Z_t K_t^{\alpha} L_t^{1-\alpha} \quad \Longrightarrow \quad (1+g) = (1+g_z)^{\frac{1}{1-\alpha}}(1+g_n)$$

Lastly, from parts (ii) and (iii) of the Balanced Growth Path definition, we can solve for the growth of per capita variables:

$$\frac{Y_{t+1}/N_{t+1}}{Y_t/N_t} = \frac{Z_{t+1}}{Z_t}\left(\frac{K_{t+1}/N_{t+1}}{K_t/N_t}\right)^{\alpha}\left(\frac{L_{t+1}/N_{t+1}}{L_t/N_t}\right)^{1-\alpha} \Longrightarrow (1+g_w) = (1+g_z)^{\frac{1}{1-\alpha}}$$

Prices: From Euler's theorem we know:

$$Y_t = \alpha Y_t + (1-\alpha)Y_t = (r_t + \delta)K_t + w_t L_t$$

Accordingly, the wage and interest rate depend on the capital-labor ratio. Growth in the capital-labor ratio is:

$$\frac{K_{t+1}/L_{t+1}}{K_t/L_t} = (1+g_z)^{\frac{1}{1-\alpha}} = 1 + g_w$$

Therefore, the growth rate for the wage is:

$$\frac{w_{t+1}}{w_t} = \frac{Z_{t+1}}{Z_t} \cdot \left(\frac{K_{t+1}/L_{t+1}}{K_t/L_t}\right)^{\alpha} = 1 + g_w$$

and the growth rate for the interest rate is:

$$\frac{r_{t+1}+\delta}{r_t+\delta} = \frac{Z_{t+1}}{Z_t} \cdot \left(\frac{K_{t+1}/L_{t+1}}{K_t/L_t}\right)^{\alpha-1} = 1$$

Therefore wages grow while interest rates do not.

Equilibrium Conditions: The detrended *asset market clearing condition* is:

$$K_t = A_t + B_t \quad \Longrightarrow \quad K = A - B$$

The detrended *resource constraint* is:

$$C_t + K_{t+1} + G_t = Y_t + (1-\delta)K_t \quad \Longrightarrow \quad C + (g+\delta)K + G = Y$$

and the detrended *government budget constraint* is:

$$G_t + rB_t = R_t + B_{t+1} - B_t \quad \Longrightarrow \quad G + (r-g)B = R$$

A.2 Individual Conditions

Preferences: We assume that labor disutility has a time-dependent component. Specifically, we assume labor disutility grows at the same rate as the utility over consumption, such that $v_{t+1}(h) = (1+g_w)^{1-\sigma} v_t(h)$. Therefore, total utility is:

$$U_t(c_t, h_t) = u(c_t) - v_t(h_t) = \left[(1+g_w)^{1-\sigma}\right]^t (u(c) - v(h)).$$

Social Security: In order for the AIME to grow at the same rate as the wage, we assume a cost of living adjustment (COLA) on Social Security taxes and payments. For social security taxes, the cap on eligible income grows at the rate of wage growth, $\bar{m}_t = (1+g_w)^t \bar{m}$. Furthermore, base payment bend points $b^{ss}_{i,t} = (1+g_w)^t b^{ss}_i$ and base payment values $\tau_{r,i,t} = (1+g_w)^t \tau_{r,i}$ for $i = 1, 2, 3$.

Tax Function: On a Balanced Growth Path, (c_t, a'_{t+1}, a_t) and $\tilde{y}_t$ must all grow at the same rate as the wage. Furthermore, the tax function must grow at the same rate as the wage. Recalling the tax function, $\Upsilon_t(\tilde{y}_t)$, τ_2 must grow at the same rate as $\tilde{y}_t^{-\tau_1}$. Rewrite as:

$$\Upsilon_t(\tilde{y}_t) = \tau_0 \left((1+g_w)^t \tilde{y} - \left([(1+g_w)^t]^{-\tau_1} \tilde{y}^{-\tau_1} + [(1+g_w)^t]^{-\tau_1} \tau_2 \right)^{-\frac{1}{\tau_1}} \right) = (1+g_w)^t \, \Upsilon(\tilde{y})$$

Individual Budget Constraint: Let the function $T(\cdot)$ contain income taxes and social security taxes or payments than an agent faces. An agent's time t budget constraint is:

$$c_t + a'_{t+1} \leq w_t \varepsilon_t h_t + (1+r_t)a_t - T_t(\cdot)$$
$$c + (1+g_w)a' \leq w\varepsilon h + (1+r)a - T(\cdot)$$

where $\{c, a', a, h, w, r, \varepsilon\}$ are stationary variables. Given that the tax function $\Upsilon(\tilde{y})$ grows at rate g_w, so will the transfer function $T(h, a, \varepsilon)$ in the infinitely lived agent model. Furthermore, given that the Social Security program $\{\bar{m}, b_i^{ss}, \tau_{r,i}\}$ grows at rate g_w, so will the transfer $T(h, a, \varepsilon, m, d)$ function in the life cycle model.

A.3 Life Cycle Model

Individual Problem: On the balanced growth path of the life cycle model, the stationary dynamic program of a working age agent is:

$$V_j(a, \varepsilon, m, 1) = \max_{c, a', h, d} \; [u(c) - v(h, d)] + [\beta\psi_j(1 + g_w)^{1-\sigma}] \sum_{\varepsilon'} \pi_j(\varepsilon'|\varepsilon) V_{j+1}(a', \varepsilon', m', d)$$

$$\begin{aligned} \text{s.t. } c + (1 + g_w)a' &\leq we(\varepsilon)h + (1 + r)(a + Tr) - \tfrac{\tau_{ss}}{2} \min\{we(\varepsilon)h, \bar{m}\} + \Upsilon(y(h, a, \varepsilon, d)) \\ a' &\geq \underline{a} \end{aligned}$$

An agent retires when they choose $d = 0$ between ages $\underline{J}_{ret}$ and $\bar{J}_{ret}$, or face mandatory retirement after age $j > \bar{J}_{ret}$. The retired agent's stationary dynamic program is:

$$V_j(a, 1, m, 0) = \max_{c, a', h} \; \left[u(c) - v(h, 0)\right] + [\beta\psi_j(1 + g_w)^{1-\sigma}] V_{j+1}(a', 1, m', 0)$$

$$\begin{aligned} \text{s.t.} \quad c + (1 + g_w)a' &\leq (1 + r)(a + Tr) + b_{ss}(m) - \Upsilon\left(r(a + Tr)\right) \\ a' &\geq \underline{a} \end{aligned}$$

Distributions: For j-th cohort at time t, the measure over $(a, \varepsilon, m, d_{-1})$ is given by:

$$\begin{aligned} \lambda_{j,t}(a_t, \varepsilon, x_t, d_{-1}) &= \lambda_{j,t-1}\left(\frac{a_t}{1 + g_w}, \varepsilon, \frac{x_t}{1 + g_w}, d_{-1}\right)(1 + g_n) \\ &= \lambda_{j,t-m}\left(\frac{a_t}{(1 + g_w)^m}, \varepsilon, \frac{x_t}{(1 + g_w)^m}, d_{-1}\right)(1 + g_n)^m \quad \forall\, m \leq t \\ &= \lambda_j(a, \varepsilon, m, d_{-1}) N_{t-j+1}. \end{aligned}$$

Therefore, $\lambda_j(a, \varepsilon, m, d_{-1})$ is a stationary distribution over age j agents that integrates to one.

Aggregation: Aggregate consumption in the life cycle model is constructed as follows.

Define the relative size of cohorts as $\mu_1 = 1$ and:

$$\mu_{j+1} = \frac{N_{t-j}}{N_t} \cdot \prod_{i=1}^{j} \psi_i = (1+g_n)^{-j} \prod_{i=1}^{j} \psi_i = \frac{\psi_j \mu_j}{1+g_n} \quad \forall\, j = 1, \ldots, J-1$$

Let $C_{j,t}$ be aggregate consumption per age-j agent, which is derived from the age-j agent's allocation:

$$C_{j,t} = \int (1+g_w)^t c_j(a, \varepsilon, m, d_{-1}) \mathbf{d}\lambda_j = (1+g_w)^t \int c_j(a, \varepsilon, m, d_{-1}) \mathbf{d}\lambda_j = (1+g_w)^t C_j$$

where C_j is the stationary aggregate consumption per age-j agent. Accordingly, aggregate consumption is:

$$\begin{aligned} C_t &= N_t \Big(C_{1,t} + \psi_1 (1+g_n)^{-1} C_{2,t} + \cdots + \left(\prod_{i=1}^{J-1} \psi_i \right) (1+g_n)^{-(J-1)} C_{J,t} \Big) \\ &= (1+g_w)^t N_t \sum_{j=1}^{J} \mu_j C_j \\ &= (1+g)^t C \end{aligned}$$

where C is the stationary level of aggregate consumption and where we have normalized $N = 1$.

We can similarly construct the remaining aggregates $\{A, K, Y, B, G\}$ on the balanced growth path. Notably, however, labor per capita does not grow. Aggregate labor per capita is constructed as:

$$L_t = N_t \sum_{j=1}^{J} \mu_j L_j \quad \Longrightarrow \quad L = \frac{L_t}{N_t} = \sum_{j=1}^{J} \mu_j \int d_j(a, \varepsilon, m, d_{-1}) \varepsilon h_j(a, \varepsilon, m, d_{-1}) \mathbf{d}\lambda_j$$

which is the sum over ages of aggregate labor per age-j agent.

A.4 Infinitely Lived Agent Model

In order to isolate the effects on optimal policy due to fundamental differences in the life cycle and infinitely lived agent models, and not due to differences in balanced growth path constructs, we want sources of output growth (e.g. TFP and population growth) to be consistent across models. Thus, we incorporate population growth into the infinitely lived agent model. To be consistent with the life cycle model, we construct a balanced growth path in which the infinitely lived agent model's income and wealth distributions grow homothetically. Our representation of this growth concept is consistent with a dynastic model in which population growth arises from agents producing offspring and valuing the utility of their offspring.

To elaborate in more detail, two additional assumptions admit a balanced growth path with population growth. First, agents exogenously reproduce at rate g_n and next period's offspring are identical to each other. Second, the parent values each offspring identically, and furthermore values each offspring as much as they value their self. Formally, if the parent has continuation value $\beta\mathbb{E}[v(a',\varepsilon')]$, then the parent values all its offspring with total value of $g_n\beta\mathbb{E}[v(a',\varepsilon')]$.

These two assumptions imply two features. First, each offspring is identical to its parent. That is, if the parent's state vector is (a',ε') next period, then so is each offspring's state vector. As a result, the value function of each offspring upon birth is $v(a',\varepsilon')$. Second, since the parent values each offspring equal to its own continuation value, it is optimal for the parent to save save $(1+g_n)a'$ in total. The portion g_na' is bequeathed to offspring, and the portion a' is kept for next period.

Individual Problem: On the balanced growth path of the Infinitely Lived Agent Model, the stationary dynamic program is:

$$v(a,\varepsilon) = \max_{c,a',h} U(c,h) + [\beta(1+g_w)^{1-\sigma}](1+g_n)\sum_{\varepsilon'}\pi(\varepsilon'|\varepsilon)v(a',\varepsilon')$$

$$\text{s.t.} \qquad c + (1+g_n)(1+g_w)a' \leq we(\varepsilon)h + (1+r)a - \Upsilon(y(h,a,\varepsilon))$$

where $y(h,a,\varepsilon) \equiv we(\varepsilon)h + ra$.

Distribution: The distribution evolves according to:

$$\lambda_{t+1}(a_{t+1},\varepsilon_{t+1}) = \sum_{\varepsilon_t}\pi(\varepsilon_{t+1}|\varepsilon_t)\int_A \mathbb{1}\left[a'_{t+1}(a_t,\varepsilon_t) = a_{t+1}\right]\lambda_t(a_t,\varepsilon_t)da_t$$

The stationary distribution $\lambda(a,\varepsilon)$ has measure 1 over $\mathcal{A}\times\mathcal{E}$ but the mass of agents grows at rate g_n:

$$\begin{aligned}
\lambda_t(a_t,\varepsilon) &= \lambda_{t-1}\left(\frac{a_t}{1+g_w},\varepsilon\right)(1+g_n) \\
&= \lambda_{t-s}\left(\frac{a_t}{(1+g_w)^s},\varepsilon\right)(1+g_n)^s \quad \forall\, s\leq t \\
&= \lambda(a,\varepsilon)N_t
\end{aligned}$$

Aggregation: To construct aggregate consumption, wealth, savings and labor, multiply individual allocations by the size of the population (N_t) and sum using the stationary distribution λ. For example, aggregate consumption is:

$$C_t = N_t \int (1+g_w)^t c(a,\varepsilon)d\lambda = (1+g)^t \int c(a,\varepsilon)d\lambda = (1+g)^t C$$

We can similarly construct the remaining aggregates $\{A, K, Y, B, G\}$ on the balanced growth path. Notably, however, aggregate labor per capita does not grow:

$$\frac{L_t}{N_t} = \int \varepsilon h(a,\varepsilon)d\lambda$$

where again $N_0 = 1$ by normalization.

B Welfare Decomposition

This appendix constructs the welfare decomposition in Section 5.3 of the main text:

$$(1+\Delta_{CEV}) = (1+\Delta_{level})(1+\Delta_{age})(1+\Delta_{distr}).$$

We will construct the three components (the levels effect, the age effect and the distribution effect) as a composite of consumption and hours effects, as follows:

$$(1+\Delta_{CEV}) = \underbrace{[(1+\Delta_{C_{level}})(1+\Delta_{H_{level}})]}_{\equiv(1+\Delta_{level})} \cdot \underbrace{[(1+\Delta_{C_{age}})(1+\Delta_{H_{age}})]}_{\equiv(1+\Delta_{age})} \cdot \underbrace{[(1+\Delta_{C_{distr}})(1+\Delta_{H_{distr}})]}_{\equiv(1+\Delta_{distr})}. \quad \text{(B1)}$$

These terms are explicitly defined in equation (B6) through equation (B11) below.

In the remainder of the section, we will first define the CEV in the context of the model. Then we will define and decompose the consumption and hours welfare effects. Finally, we will verify that the decomposition in fact holds.

B.1 Preliminaries

Consider two economies, $i \in \{1, 2\}$. Define ex ante welfare in economy $i \in \{1, 2\}$ derived from consumption, hours and retirement allocations $\{c^i_j(\mathbf{s}), h^i_j(\mathbf{s}), d^i_j(\mathbf{s})\}_{j=1}^J$ over states $\mathbf{s} \equiv (a, \varepsilon, m, d_{-1})$ distributed with $\lambda^i_j(\mathbf{s})$ as:

$$S^i = U(c^i) - V^h(h^i) - V^d(d^i)$$

where

$$U(c^i) \equiv \int \mathbb{E}_0 \left[\sum_{j=1}^{J} \beta^{j-1} \psi_j u\left(c_j^i\right) \right] \mathbf{d}\lambda_1^i$$

$$V^h(h^i) \equiv \int \mathbb{E}_0 \left[\sum_{j=1}^{J} \beta^{j-1} \psi_j v\left(h_j^i\right) \right] \mathbf{d}\lambda_1^i$$

$$V^d(d^i) \equiv \int \mathbb{E}_0 \left[\sum_{j=1}^{J} \beta^{j-1} \psi_j \, \chi_2 \, d_j^i \right] \mathbf{d}\lambda_1^i.$$

Denote the Consumption Equivalent Variation (CEV) by Δ_{CEV}, which is defined as the percent of expected lifetime consumption that an agent inhabiting economy $i = 1$ would pay *ex ante* in order to inhabit economy $i = 2$:

$$(1 + \Delta_{CEV})^{1-\sigma} U(c^1) - V^h(h^1) - V^d(d^1) = U(c^2) - V^h(h^2) - V^d(d^2). \tag{B2}$$

B.2 Consumption Effect Decomposition

First we decompose the CEV into levels, age and distribution effects for consumption allocations. The overall consumption effect is:

$$(1 + \Delta_C)^{1-\sigma} U(c^1) - V^h(h^1) - V^d(d^1) = U(c^2) - V^h(h^1) - V^d(d^1). \tag{B3}$$

For the level effect, note that aggregate consumption in economy i is

$$C^i = \sum_{j=1}^{J} \mu_j \int_{\mathbf{S}} c_j^i(\mathbf{s}) \mathbf{d}\lambda_j^i(\mathbf{s}).$$

We follow Conesa et al. (2009) in defining the level effect, in a different but equivalent way to Floden (2001), by

$$(1 + \Delta_{C_{level}})^{1-\sigma} U(c^1) - V^h(h^1) - V^d(d^1) = U(\left(C^2/C^1\right) c^1) - V^h(h^1) - V^d(d^1),$$

or more succinctly,

$$(1 + \Delta_{C_{level}}) = \frac{C^2}{C^1}.$$

For the age effect, note that age-specific average consumption in economy i is

$$C_j^i = \int_{\mathbf{S}} c_j^i(\mathbf{s}) \mathbf{d}\lambda_j^i(\mathbf{s}),$$

and utility over age-cohort average level of consumption at each age is,

$$U(C_j^i) \equiv \int \mathbb{E}_0 \left[\sum_{j=1}^{J} \beta^{j-1} \psi_j \int u\left(C_j^i\right) \right] \mathbf{d}\lambda_1^i.$$

Then define the consumption age effect in economy i by

$$U\left((1-\omega_{C_{age}}^i)C^i\right) = U(C_j^i),$$

such that

$$(1-\omega_{C_{age}}^i) = \left(\frac{\sum_{j=1}^{J} \beta^{j-1} \psi_j u\left(C_j^i\right)}{\left[\sum_{j=1}^{J} \beta^{j-1} \psi_j\right] u(C^i)} \right)^{\frac{1}{1-\sigma}},$$

which gives the overall consumption age effect,

$$(1+\Delta_{C_{age}}) \equiv \frac{1-\omega_{C_{age}}^2}{1-\omega_{C_{age}}^1} = \frac{\left(\sum_{j=1}^{J} \beta^{j-1} \psi_j u\left(C_j^2\right)\right)^{\frac{1}{1-\sigma}} / C^2}{\left(\sum_{j=1}^{J} \beta^{j-1} \psi_j u\left(C_j^1\right)\right)^{\frac{1}{1-\sigma}} / C^1}.$$

Lastly, again following Floden (2001) and Conesa et al. (2009), define the consumption distribution effect in economy i as the residual of the overall consumption effect:

$$U\left((1-\omega_{distr}^i)C_j^i\right) = U\left(c^i\right)$$

such that

$$(1-\omega_{C_{distr}}^i) = \left(\frac{\sum_{j=1}^{J} \beta^{j-1} \psi_j u(c^i)}{\sum_{j=1}^{J} \beta^{j-1} \psi_j u(C_j^i)} \right)^{\frac{1}{1-\sigma}},$$

which gives the overall consumption distribution effect,

$$(1+\Delta_{C_{distr}}) \equiv \frac{1-\omega_{C_{distr}}^2}{1-\omega_{C_{distr}}^1} = \frac{\left(U(c^2)/U(C_j^2)\right)^{\frac{1}{1-\sigma}}}{\left(U(c^1)/U(C_j^1)\right)^{\frac{1}{1-\sigma}}}.$$

B.3 Hours Effect Decomposition

Likewise we define the overall hours effect by

$$(1+\Delta_H)^{1-\sigma}U(c^2) - V^h(h^1) - V^d(d^1) = U(c^2) - V^h(h^2) - V^d(d^2). \tag{B4}$$

For the level effect, note that aggregate hours and the mass of working agents in economy i is

$$H^i = \sum_{j=1}^{J} \mu_j \int_{\mathbf{S}} h_j^i(\mathbf{s}) \mathbf{d}\lambda_j^i(\mathbf{s}),$$

$$I^i = \sum_{j=1}^{J} \mu_j \int_{\mathbf{S}} d_j^i(\mathbf{s}) \mathbf{d}\lambda_j^i(\mathbf{s}).$$

We follow Conesa et al. (2009) in defining the hours level effect. However, since our economy features both an intensive and extensive margin labor decision, we simultaneously decompose welfare arising from hours and retirement decisions,

$$(1+\Delta_{H_{level}})^{1-\sigma}U(c^2) - V^h(h^1) - V^d(d^1) = U(c^2) - V^h\left(\frac{H_2}{H_1}h^1\right) - V^d\left(\frac{I^2}{I^1}d^1\right)$$

For the age effect, note that age-specific average hours and average mass of working agents in economy i is

$$H_j^i = \int_{\mathbf{S}} h_j^i(\mathbf{s}) \mathbf{d}\lambda_j^i(\mathbf{s}),$$

$$I_j^i = \int_{\mathbf{S}} d_j^i(\mathbf{s}) \mathbf{d}\lambda_j^i(\mathbf{s}),$$

and expected lifetime utility over age-cohort average level of hours and working at each age is,

$$V^h(H_j^i) \equiv \int \mathbb{E}_0 \left[\sum_{j=1}^{J} \beta^{j-1}\psi_j v\left(H_j^i\right)\right] \mathbf{d}\lambda_1^i,$$

$$V^d(I_j^i) \equiv \int \mathbb{E}_0 \left[\sum_{j=1}^{J} \beta^{j-1}\psi_j \chi_2 I_j^i\right] \mathbf{d}\lambda_1^i,$$

Then define the hours age effect in economy i by

$$(1+\Delta_{H_{age}})^{1-\sigma}U(c^2) - V^h\left(\frac{H_2}{H_1}h^1\right) - V^d\left(\frac{I^2}{I^1}d^1\right) = U(c^2) - V^h\left(\frac{V^h(H_j^2)}{V^h(H_j^1)}h^1\right) - V^d\left(\frac{V^d(I_j^2)}{V^d(I_j^1)}d^1\right)$$

Following Floden (2001) and Conesa et al. (2009), the hours distribution effect is then a residual of the overall hours effect:

$$(1+\Delta_{H_{distr}})^{1-\sigma}U(c^2) - V^h\left(\frac{V^h(H_j^2)}{V^h(H_j^1)}h^1\right) - V^d\left(\frac{V^d(I_j^2)}{V^d(I_j^1)}d^1\right) = U(c^2) - V^h(h^2) - V^d(d^2)$$

B.4 Verification of Decomposition

Overall CEV: From equation (B2), the CEV can be rewritten as:

$$(1+\Delta_{CEV}) = \left(\frac{\left(U(c^2) - V^h(h^2) - V^d(d^2)\right) + V^h(h^1) + V^d(d^1)}{U(c^1)}\right)^{\frac{1}{1-\sigma}}. \tag{B5}$$

Consumption Effect: From equation (B3), the consumption effect can be rewritten as:

$$(1+\Delta_C) = \left(\frac{U(c^2)}{U(c^1)}\right)^{\frac{1}{1-\sigma}}.$$

Explicitly define each component of the consumption welfare decomposition as follows:

$$(1+\Delta_{C_{level}}) = \frac{C^2}{C^1} \tag{B6}$$

$$(1+\Delta_{C_{age}}) = \frac{(U(C_j^2)/U(C_j^1))^{\frac{1}{1-\sigma}}}{C^2/C^1} \tag{B7}$$

$$(1+\Delta_{C_{distr}}) = \frac{(U(c^2)/U(c^1))^{\frac{1}{1-\sigma}}}{(U(C_j^2)/U(C_j^1))^{\frac{1}{1-\sigma}}} \tag{B8}$$

Accordingly, the consumption effect decomposition is verified as follows,

$$
\begin{aligned}
(1+\Delta_C) &= (1+\Delta_{C_{level}}) \quad \cdot \quad (1+\Delta_{C_{age}}) \quad \cdot \quad (1+\Delta_{C_{distr}}) \\
&= (C^2/C^1) \quad \cdot \quad \frac{(U(C_j^2)/U(C_j^1))^{\frac{1}{1-\sigma}}}{C^2/C^1} \quad \cdot \quad \frac{(U(c^2)/U(c^1))^{\frac{1}{1-\sigma}}}{(U(C_j^2)/U(C_j^1))^{\frac{1}{1-\sigma}}} \\
&\stackrel{\checkmark}{=} \left(\frac{U(c^2)}{U(c^1)}\right)^{\frac{1}{1-\sigma}}
\end{aligned}
$$

Hours Effect: From equation (B4), the hours effect can be rewritten as:

$$(1+\Delta_H) = \left(\frac{U(c^2) - V^h(h^2) - V^d(d^2) + V^h(h^1) + V^d(d^1)}{U(c^2)}\right)^{\frac{1}{1-\sigma}}$$

It is easy to show that:

$$(1+\Delta_H) = \frac{1+\Delta_{CEV}}{1+\Delta_C}.$$

Next, we wish to show that the hours effect decomposition holds:

$$(1+\Delta_H) = (1+\Delta_{H_{level}})(1+\Delta_{H_{age}})(1+\Delta_{H_{distr}})$$

Explicitly define each component of the hours welfare decomposition as follows:

$$(1+\Delta_{H_{level}}) = \left[1 + \left(1 - \left(\frac{H^2}{H^1}\right)^{1+\frac{1}{\gamma}}\right)\frac{V^h(h^1)}{U(c^2)} + \left(1 - \frac{I^2}{I^1}\right)\frac{V^d(d^1)}{U(c^2)}\right]^{\frac{1}{1-\sigma}} \tag{B9}$$

$$(1+\Delta_{H_{age}}) = \left[1 + \left(\left(\frac{H^2}{H^1}\right)^{1+\frac{1}{\gamma}} - \left(\frac{V^h(H_j^2)}{V^h(H_j^1)}\right)^{1+\frac{1}{\gamma}}\right)\frac{V^h(h^1)}{U(c^2)} + \left(\frac{I^2}{I^1} - \frac{V^d(I_j^2)}{V^d(I_j^1)}\right)\frac{V^d(d^1)}{U(c^2)}\right]^{\frac{1}{1-\sigma}} \tag{B10}$$

$$(1+\Delta_{H_{distr}}) = \left[1 + \left(\frac{V^h(H_j^2)}{V^h(H_j^1)}\right)^{1+\frac{1}{\gamma}}\frac{V^h(h^1)}{U(c^2)} - \frac{V^h(h^2)}{U(c^2)} + \left(\frac{V^d(I_j^2)}{V^d(I_j^1)}\right)\frac{V^d(d^1)}{U(c^2)} - \frac{V^d(d^2)}{U(c^2)}\right]^{\frac{1}{1-\sigma}} \tag{B11}$$

The decomposition can be verified using a first order approximation of the $i = 2$ allocation around the $i = 1$ allocation and therefore a first order approximation of $\Delta_{H_{level}}, \Delta_{H_{age}}, \Delta_{H_{distr}}$ around zero. Noting that $u'(c)c/u(c) = (1-\sigma)$ and $v'(h)h/v(h) = 1 + 1/\gamma$, the first order approximations yield the following expressions for the hours welfare decomposition:

$$\Delta_H \approx \frac{1}{1-\sigma}\left(\frac{U(c^2) - V^h(h^2) - V^d(d^2) + V^h(h^1) + V^d(d^1)}{U(c^2)} - 1\right)$$

$$\Delta_{H_{level}} \approx \frac{1}{1-\sigma}\left(\left(1 - \left(1+\frac{1}{\gamma}\right)\frac{H^2}{H^1}\right)\frac{V^h(h^1)}{U(c^2)} + \left(1 - \frac{I^2}{I^1}\right)\frac{V^d(d^1)}{U(c^2)}\right)$$

$$\Delta_{H_{age}} \approx \left(\frac{1+\frac{1}{\gamma}}{1-\sigma}\right)\left(\frac{H^2}{H^1} - \frac{V^h(H_j^2)}{V^h(H_j^1)}\right)\frac{V^h(h^1)}{U(c^2)} + \frac{1}{1-\sigma}\left(\frac{I^2}{I^1} - \frac{V^d(I_j^2)}{V^d(I_j^1)}\right)\frac{V^d(d^1)}{U(c^2)}$$

$$\Delta_{H_{distr}} \approx \frac{1}{1-\sigma}\left(\left(1+\frac{1}{\gamma}\right)\left(\frac{V^h(H_j^2)}{V^h(H_j^1)}\right)\frac{V^h(h^1)}{U(c^2)} - \frac{V^h(h^2)}{U(c^2)} + \left(\frac{V^d(I_j^2)}{V^d(I_j^1)}\right)\frac{V^d(d^1)}{U(c^2)} - \frac{V^d(d^2)}{U(c^2)}\right)$$

Since $\log(1+\Delta) \approx \Delta$,

$$\log(1+\Delta_H) = \log(1+\Delta_{H_{level}}) + \log(1+\Delta_{H_{age}}) + \log(1+\Delta_{H_{distr}})$$

implies

$$\Delta_H \approx \Delta_{H_{level}} + \Delta_{H_{age}} + \Delta_{H_{distr}}.$$

The approximation above can be directly verified.