Valuing Private Equity Investments Strip by Strip*

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Abstract

We propose a new valuation method for private equity investments. First, we construct a cash-flow replicating portfolio for the private investment, using cash-flows on various listed equity and fixed income instruments. The second step values the replicating portfolio using a flexible asset pricing model that accurately prices the systematic risk in listed equity and fixed income instruments of different horizons. The method delivers a measure of the risk-adjusted profit earned on a PE investment, a time series for the expected return on PE fund categories, and a time series for the residual net asset value in a fund. We apply the method to real estate, infrastructure, buyout, and venture capital funds, and find modestly positive average risk-adjusted profits with substantial cross-sectional variation, and declining expected returns in the later part of the sample.

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1 Introduction

Private equity investments have risen in importance over the past twenty-five years, relative to public equity. Indeed, the number of publicly listed firms has been falling since 1997, especially among smaller firms. Private equity funds account for $4.7 trillion in assets under management, of which real estate funds comprise $800 billion (Preqin). Large institutional investors now allocate substantial fractions of their portfolios to such alternative investments. For example, the celebrated Yale University endowment has a portfolio weight of over 50% in alternative investments. Pension funds and sovereign wealth funds have also ramped up their allocations to alternatives. As the fraction of overall wealth that is held in the form of private investment grows, so does the importance of developing appropriate valuation methods. The non-traded nature of the assets and their irregular cash-flows makes this a challenging problem.

As with any investment, the value of a private equity (PE) investment equals the present discounted value of its cash-flows. The general partner (GP, fund manager) deploys the capital committed by the limited partners (LPs, investors) by investing in a portfolio of risky projects. The risky projects may pay some interim cash-flows that are distributed back to the LPs. The bulk of the cash flows arise when the GP sells the projects, and distributes the proceeds net of fees (carry, promote) to the LPs. The main challenge in evaluating a PE investment is how to adjust the profits the LP earns for the systematic risk inherent in the PE cash flows. Industry practice is to report the ratio of distributions to capital contributions (TVPI) and/or the internal rate of return (IRR). Neither metric takes into account the riskiness of the cash-flows.

We propose a novel two-step methodology that centers on the nature and the timing of cash-flow risk for PE investments. In a first step, we estimate the exposure of PE funds' cash-flows to the cash-flows of a set of publicly listed securities. The main analysis considers Treasury bonds, the aggregate stock market, a real estate stock index (REIT), an infrastructure stock index, small stocks, growth stocks, and natural resource stocks as the listed securities. The method considers a much richer cross-section of risks than the literature hitherto, and easily accommodates additional publicly-traded risk factors. Inspired by Lettau and Wachter (2011) and van Binsbergen, Brandt, and Koijen (2012), we "strip" the sequence of PE cash-flows horizon by horizon, and estimate factor exposures to the corresponding listed strip cash flows. Our identification assumes that the systematic cash-flow exposures depend on PE category, horizon, and the underlying market conditions at the time of fund origination, as proxied by the price-dividend ratio on the stock market. All PE funds within the same category and vintage have the same exposures to the asset strips. We estimate this exposure both using OLS as well as using a lasso estimation which shrinks the cross-section and enforces only positive coefficients (corresponding to a long-only replicating portfolio).

In a second step, we use a rich, no-arbitrage asset pricing model that prices the asset strips. This is necessary since data on prices for dividend strips are not available for any asset besides the aggregate stock market, and even then only for a small fraction of the sample. Strips on REITs, infrastructure, or any other equity factor are unavailable. The model estimates the prices of risk by closely matching the history of bond yields of different maturities, as well as prices and expected returns on the five equity indices. It also matches the risk premium on short-maturity dividend futures from 2003-2014, calculated in the data by van Binsbergen, Hueskes, Koijen, and Vrugt (2013), and van Binsbergen and Koijen (2017), and the time series of the price-dividend ratio on 2-year cumulative dividend strips and the share of the overall stock market they represent from 1996-2009 as backed out from options data by van Binsbergen, Brandt, and Koijen (2012). With the market price of risk estimates in hand, we can price risky cash-flows at each date and at each horizon in the bond, aggregate stock, small stock, growth stock, REIT, natural resource, and infrastructure markets. We use the shock price elasticities of Borovička and Hansen (2014) to understand how risk prices change with horizon in the model.

Combining the cash-flow replicating portfolio of strips obtained from the first step with the prices for these strips from the asset pricing model estimated in the second step, we obtain the fair price for the PE-replicating portfolio in each vintage and category. Each PE investment in the data is scaled to represent $1 of capital committed. Therefore, the replicating portfolio of strips must deploy the same $1 of capital. This budget feasibility constraint on the replicating portfolio directly affects PE fund performance evaluation. A time of high strip prices is a time when the replicating portfolio must buy fewer strips. PE funds started at that time (i.e., of that vintage) are more likely to have cash flows that exceed those on the replicating portfolio, all else equal. Of course, the assets that PE funds acquire may be more expensive as well, so that out-performance is an empirical question. The risk-adjusted profit (RAP) of an individual PE fund is the net present value of the excess cash flows, the difference between the realized cash flows on the PE fund and the realized cash flows on the replicating portfolio in that vintage-category. Under the joint null hypothesis of the correct asset pricing model and the absence of (asset selection or market timing) skill, the RAP is zero.

The asset pricing model is used also to compute the expected return on a PE investment, which reflects the systematic risk exposure of the PE fund. Our method breaks down the expected return into its various horizon components (strips), and, at each horizon, into its exposures to the various risk factors. Since the expected return on the listed strips varies with the state of the economy, so does the expected return on PE investments. Our method can also be used to ask what the expected return is on all outstanding PE investments, by aggregating across current vintages. The method can further be used to calculate the residual net asset value (NAV) of PE funds at each point during their life cycle, providing an alternative measure to the NAV reported by funds themselves. Finally, by providing the expected return on PE, and the covariances of PE funds with traded securities, our approach facilitates portfolio analysis with alternatives for which return time series are unavailable.

In the absence of a full menu of dividend strips (e.g., REIT strips or small growth stock strips are not currently available), our results imply that PE funds are a vehicle to provide an investor with exposure to short- and medium-horizon risk in real estate, infrastructure, or small growth markets. Access to such exposure is becoming increasingly important in light of the decline in the share of publicly listed investments, in particular in the small growth space, and in real estate and infrastructure markets.

We use data from Preqin on all PE funds with non-missing cash-flow information that were started between 1980 and 2017. Cash-flow data until December 2017 are used in the analysis. Our sample includes 4,219 funds in seven investment categories. The largest categories are Buyout, Real Estate, and Venture Capital. The main text reports results for these three categories as well as Infrastructure, and relegates the results for the other three to the appendix. The PE data from Preqin are subject to the usual selection bias concerns.[1]

Our key finding is that the risk-adjusted profit (RAP) to investors in PE funds exhibits strong cross-sectional variation. We find average RAP around 20-30 cents per $1 invested in the Buyout category, for instance. While PE cash-flows have significant exposures to several public market risk factors, the market prices of these exposures are high. PE funds therefore offer investors access to these exposures at a cost that is generally below that in public stock and bond markets.

However, these estimates are sensitive to the choice of cross-sectional factors. We

[1] Preqin data are thought to understate performance. Some high-performing funds that are closed to outside investors to protect from FOIA requests are not in our data set. An alternative data set provided by Burgiss has superior coverage of these funds (Brown, Harris, Jenkinson, Kaplan, and Robinson, 2015). Preliminary results indicate that our main findings continue to hold in this data set.

find that a substantial component of the profits in Buyout, Real Estate, and Infrastructure funds can be instead attributed to compensation for factor exposure. Our estimates of risk-adjusted profit are considerably lower when taking this factor loading into account. By contrast, our estimates of risk-adjusted profit in the Venture Capital category substantially improve when taking into account that a better estimated replicating portfolio would counterfactually invest in factors which look like capital gains on growth stocks. These profits are typically decreasing over time. In the latest vintages in the Buyout and VC categories, we observe close to zero risk-adjusted profits in our preferred lasso estimation.

Related Literature This paper contributes to a large empirical literature on performance evaluation in private equity funds, such as Kaplan and Schoar (2005), Cochrane (2005), Korteweg and Sorensen (2017), Harris, Jenkinson, and Kaplan (2014), Phalippou and Gottschalg (2009), Robinson and Sensoy (2011), among many others. Most of this literature focuses on Buyout and Venture Capital funds, though recent work in valuing privately-held real estate assets includes Peng (2016) and Sagi (2017). Ammar and Eling (2015) have studied infrastructure investments. This literature has found mixed results regarding PE fund outperformance and persistence thereof, depending on the data set and period in question. Our replicating portfolio approach results in a substantially positive estimate of risk-adjusted profits for PE funds, with large cross-sectional and time-series variation.

While performance evaluation in private equity is still often expressed as an internal rate of return or a ratio of distributions to capital committed, several important papers have incorporated risk into the analysis. The public market equivalent (PME) approach of Kaplan and Schoar (2005) compares the private equity investment to a public market benchmark (the aggregate stock market) with the same magnitude and timing of cashflows. Sorensen and Jagannathan (2015) assess the PME approach from a SDF perspective. The closest antecedent to our paper is Korteweg and Nagel (2016), who propose a generalized PME approach (GPME) that relaxes the assumption that the beta of PE funds to the stock market is one. This is particularly important in their application to VC funds. Like ours, the PME and GPME approaches avoid making strong assumptions on the return-generating process of the PE fund, because they work directly with the cashflows. Cochrane (2005) and Korteweg and Sorensen (2010) discuss this distinction. In contrast, much of the literature assumes linear beta-pricing relationships, e.g., Ljungqvist and Richardson (2003), Driessen, Lin, and Phalippou (2012).

The literature that estimates beta exposures of PE funds with respect to the stock market has generally estimated stock market exposures of Buyout funds above one and even higher estimates for VC funds. E.g., Gompers and Lerner (1997); Ewens, Jones, and Rhodes-Kropf (2013); Peng (2001); Woodward (2009); Korteweg and Nagel (2016). Our work contributes to this literature by allowing for a flexible estimation approach across horizon and vintage, for risk exposure estimates to differ by category, by considering a broader set of PE categories than typically examined, and especially by going beyond the aggregate stock market as the only source of aggregate risk. VC funds are found to load on small stock and growth stock risk. Results for VC funds have implications for the returns on entrepreneurial activity (Moskowitz and Vissing-Jorgensen, 2002). Finally, we connect the systematic risk exposures of funds to a rich asset pricing model, which allows us to estimate risk-adjusted profits and time-varying expected returns.

Like Korteweg and Nagel (2016), we estimate a stochastic discount factor (SDF) from public securities. Our SDF contains additional risk factors and richer price-of-risk dynamics. Those dynamics are important for generating realistic, time-varying risk premia on bond and stock strips, which are the building blocks of our PE valuation method. The SDF model extends earlier work by Lustig, Van Nieuwerburgh, and Verdelhan (2013) who value a claim to aggregate consumption to help guide the construction of consumption-based asset pricing models. The asset pricing model combines a vector auto-regression model for the state variables as in Campbell (1991, 1993, 1996) with a no-arbitrage model for the (SDF) as in Duffie and Kan (1996); Dai and Singleton (2000); Ang and Piazzesi (2003). The SDF model needs to encompass the sources of aggregate risk that the investor has access to in public securities markets and that PE funds are exposed to. The question of performance evaluation then becomes whether, at the margin, PE funds add value to a portfolio that already contains these traded assets.

In complementary work, Ang, Chen, Goetzmann, and Phalippou (2017) filter a time series of realized private equity returns using Bayesian methods. They then decompose that time series into a systematic component, which reflects compensation for factor risk exposure, and an idiosyncratic component (alpha). While our approach does not recover a time series of realized PE returns, it does recover a time series of *expected* PE returns. At each point in time, the asset pricing model can be used to revalue the replicating portfolio for the PE fund. Since it does not require a difficult Bayesian estimation step, our approach is more flexible in terms of number of factors as well as the factor risk premium dynamics. Other important methodological contributions to the valuation of private equity include Driessen, Lin, and Phalippou (2012), Sorensen, Wang, and Yang (2014), and Metrick and Yasuda (2010).

The rest of the paper is organized as follows. Section 2 describes our methodology. Section 3 sets up and solves the asset pricing model. Section 4 presents the main results on the risk-adjusted profits and expected returns of PE funds. Section 5 concludes. The Appendix provides derivations and additional results.

2 Methodology

PE investments are finite-horizon strategies, typically around ten to fifteen years in duration. Upon inception of the fund, the investor (LP) commits capital to the fund manager (GP). The GP deploys that capital at his discretion, but typically within the first 2-4 years. Intermediate cash-flows may accrue from the operations of the assets, for example, net operating income from renting out an office building. Towards the end of the life of the fund (typically in years 5-10), the GP "harvests" the assets and distributes the proceeds to the investor after subtracting fees (including the carry or promote). These distribution cash-flows are risky, and understanding (and pricing) the nature of the risk in these cash-flows is the key question in this paper.

Denote the sequence of net-of-fee cash-flow distributions for fund i by $\{X^i_{t+h}\}_{h=0}^T$. Time t is the inception quarter of the fund, the vintage. The horizon h indicates the number of quarters since fund inception. We allocate all funds started in the same calendar year to the same vintage. The maximum horizon H is set to 60 quarters to allow for "zombie" funds that continue past their projected life span of approximately 10 years. All cash flows observed after quarter H are allocated to quarter $H+1$. Monthly fund cash-flows are aggregated to the quarterly frequency. All PE cash-flows in our data are reported for a \$1 investor commitment.

Once the capital is committed, the GP has discretion to call that capital. We take the perspective that the risk-adjusted profit (RAP) measure should credit the GP for the skillful timing of capital calls. Correspondingly, we assume that the replicating portfolio is fully invested at time t. If strategic delay in capital deployment results in better investment performance, the RAP will reflect this. In sum, we purposely do not use the data on capital calls, only the distribution cash flow data.[2]

[2]Note that under this assumption, the net present value of deployed capital may differ from \$1. Our

2.1 Two-Step Approach

In a first step, we use our asset pricing model, spelled out in the next section, to price the zero coupon bond and equity strips. Let the $HK \times 1$ vector $\boldsymbol{F}_{t,t+h}$ be the vector of cash flows on the securities in the replicating portfolio. The first H elements of $\boldsymbol{F}_{t,t+h}$ are constant equal to 1. They are the cash-flows on nominal zero-coupon U.S. Treasury bonds that pay \$1 at time $t+h$. The other $H(K-1)$ elements of $\boldsymbol{F}_{t,t+h}$ denote risky cash-flow realizations at time $t+h$. They are the payoffs on "zero coupon equity" or "dividend strips" (Lettau and Wachter, 2011; van Binsbergen, Brandt, and Koijen, 2012). They pay one (risky) cash-flow at time $t+h$ and nothing at any other date. We scale the risky dividend at $t+h$ by the cash flow at fund inception time t. For example, a risky cash-flow of $F_{t,t+h}(k) = \frac{D_{t+h}(k)}{D_t(k)} = 1.05$ implies that there was a 5% realized cash-flow growth rate between periods t and $t+h$ on the k^{th} asset in the replicating portfolio. This scaling gives the strips a "face value" around 1, comparable to that of the zero coupon bond, which makes bond and stock exposures comparable in magnitude.

In addition to dividend strips, we also allow capital gain strips, or gain strips for short, to enter the PE-replicating portfolio. A gain strip bought at time t pays one cash flow at time $t+h$ equal to the realized ex-dividend price of the stock. We scale this cash flow by the current stock price. For example, a 20-quarter gain strip on the aggregate stock market bought at time t pays the single cash flow $P^m_{t+20}/P^m_t = 1.05$ at time $t+20$, equal to the 5% cumulative capital gain on the stock market over the 20-quarter investment horizon. By value additivity, the stock price at time t ($t+h$) is the sum of the prices on all dividend strips after time t ($t+h$). Therefore, the price of a gain strip will usually be smaller than 1. The reason for including gain strips in the replicating portfolio is that PE cash flows during the harvesting period are likely to reflect asset dispositions. These dispositions take place at prices that reflect all future cash flows on those assets. It is conceivable that these late-life distributions are more highly correlated with publicly listed prices rather than dividends. We let $\boldsymbol{F}_{t,t+h}$ contain both dividend and gain strip cash flow realizations.

Denote the $HK \times 1$ price vector for strips by $\boldsymbol{P}_{t,h}$. The first $H \times 1$ elements of this price vector are the prices of nominal zero-coupon bonds of maturities $h = 1, \cdots, H$, which we denote by $\boldsymbol{P}_{t,h}(1) = \boldsymbol{P}^{\$}_{t,h}$. Let the one-period stochastic discount factor (SDF) be M_{t+1},

methodology can handle capital calls. In that case, the replicating portfolio would mimic not only the distribution cash flows but also the call cash flows. The calls would be treated as negative bond strip positions.

then the h-period SDF is:

$$M_{t,t+h} = \prod_{j=1}^{h} M_{t+j}.$$

The (vector of) strip prices satisfy the (system of) Euler equation:

$$\boldsymbol{P}_{t,h} = \mathbb{E}_t[M_{t,t+h}\boldsymbol{F}_{t,t+h}].$$

Strip prices reflect expectations of the SDF rather than realizations.

The second step of our approach is to obtain the cash-flow replicating portfolio of strips for the PE cash-flow distributions. Denote the cash flow on the replicating portfolio by $\boldsymbol{\beta}^i_{t,h}\boldsymbol{F}_{t,t+h}$, where the $1 \times HK$ vector $\boldsymbol{\beta}^i_{t,h}$ denotes the exposure of PE fund i to the HK assets in the replicating portfolio. We estimate the exposures from a projection of cash-flows realized at time $t+h$ of PE funds started at time t on the cash-flows of the risk-free and risky strips:

$$X^i_{t+h} = \boldsymbol{\beta}^i_{t,h}\boldsymbol{F}_{t,t+h} + e^i_{t+h}. \tag{1}$$

where e denotes the idiosyncratic cash-flow component, orthogonal to $F_{t,t+h}$. The vector $\beta^i_{t,h}$ describes how many units of each strip are in the replicating portfolio for the fund cash-flows. Equation (1) is estimated on a sample of all funds in a given category, all vintages t, and all horizons h. We impose cross-equation restrictions on this estimation, as explained below.

Budget Feasibility We use the asset pricing model to ensure that the replicating portfolio of bond and stock strips for the PE fund is *budget feasible*. The portfolio of strips must cost exactly \$1, the same initial outlay as for the PE investment. The replicating portfolio estimated from equation (1) does not automatically satisfy budget feasibility. We define a $1 \times HK$ vector of rescaled portfolio positions, $\boldsymbol{q}^i$, that costs exactly \$1:

$$\boldsymbol{q}^i_{t,h} = \frac{\boldsymbol{\beta}^i_{t,h}}{\sum_{h=1}^{H}\boldsymbol{\beta}^i_{t,h}\boldsymbol{P}_{t,h}} \quad \Rightarrow \quad \sum_{h=1}^{H}\boldsymbol{q}^i_{t,h}\boldsymbol{P}_{t,h} = 1.$$

The strip prices $\boldsymbol{P}_{t,h}$ are given by the asset pricing model. This is the first place we use the asset pricing model. Since the strip prices change over time, each vintage has its own rescaling. This induces time variation in the replicating portfolio, adding to the time variation coming from $\boldsymbol{\beta}^i_{t,h}$.

With the budget feasible replicating portfolio in hand, we redefine the idiosyncratic component of fund cash-flows as v^i:

$$v^i_{t+h} = X^i_{t+h} - \boldsymbol{q}^i_{t,h}\boldsymbol{F}_{t,t+h}.$$

Under the joint null hypothesis of the asset pricing model and no fund outperformance, the expected present discounted value of fund cash-flow distributions must equal the $1 initially paid in by the investor:

$$\mathbb{E}_t\left[\sum_{h=1}^{H} M_{t,t+h}X^i_{t+h}\right] = \mathbb{E}_t\left[\sum_{h=1}^{H} M_{t,t+h}\boldsymbol{q}^i_{t,h}\boldsymbol{F}_{t,t+h}\right] = \sum_{h=1}^{H}\boldsymbol{q}^i_{t,h}\boldsymbol{P}_{t,h} = 1, \tag{2}$$

where the first equality follows from the fact that the idiosyncratic cash-flow component v^i is uncorrelated with the SDF since all priced cash-flow shocks are included in the vector F under the null hypothesis.

Expected Returns The second place where we use the asset pricing model is to calculate the expected return on the PE investment over the life of the investment. It equals the expected return on the replicating portfolio of strips:

$$\mathbb{E}_t\left[R^i\right] = \sum_{h=1}^{H}\sum_{k=1}^{K}\boldsymbol{w}^i_{t,h}(k)\mathbb{E}_t\left[\boldsymbol{R}_{t+h}(k)\right] \tag{3}$$

where $\boldsymbol{w}^i$ is a $1 \times HK$ vector of replicating portfolio weights with generic element $w^i_{t,h}(k) = q^i_{t,h}(k)P_{t,h}(k)$. The $HK \times 1$ vector $\mathbb{E}_t[\boldsymbol{R}]$ denotes the expected returns on the K traded asset strips at each horizon h. The asset pricing model provides the expected returns on these strips. Equation (3) decomposes the risk premium into compensation for exposure to the various risk factors, horizon by horizon. The expected return is measured over the life of the fund (not annualized). It can be annualized by taking into account the maturity of the fund. Akin to a MacCauley duration, we define the maturity of the fund, expressed in years (rather than quarters), as:

$$\delta^i_t = \frac{1}{4}\sum_{h=1}^{H}\sum_{k=1}^{K}\boldsymbol{w}^i_{t,h}(k)h \tag{4}$$

The annualized expected fund return is then:

$$\mathbb{E}_t\left[R^i_{ann}\right] = \left(1 + \mathbb{E}_t\left[R^i\right]\right)^{1/\delta^i_t} - 1 \tag{5}$$

Risk-Adjusted Profit Performance evaluation of PE funds requires quantifying the LP's profit on a particular PE investment, after taking into account its riskiness. This ex-post realized, risk-adjusted profit is the second main object of interest. Under the maintained assumption that all the relevant sources of systematic risk are captured by the replicating portfolio, the PE cash-flows consist of one component that reflects compensation for risk and a risk-adjusted profit (RAP) equal to the discounted value of the idiosyncratic cash-flow component. The latter component for fund i in vintage t equals:

$$RAP_t^i = \sum_{h=1}^{H} P_{t,h}^{\$} v_{t+h}^i \tag{6}$$

Since the idiosyncratic cash-flow components are orthogonal to the priced cash-flow shocks, they are discounted at the risk-free interest rate. Since the term structure of risk-free bond prices $P_{t,h}^{\$}$ is known at time t, there is no measurement error involved in the discounting. The null hypothesis of no outperformance is $\mathbb{E}[RAP_t^i] = 0$, where the expectation is taken on average across funds.

A fund with strong asset selection skills picks investment projects with payoffs superior to the payoffs on traded assets and will have a positive RAP. Additionally, a fund with market timing skills, which invests at the right time (within the investment period) and sells at the right time (within the harvesting period) will have positive risk-adjusted profit.[3] Alternatively, if capital lock-up in the PE fund structure enables managers to earn an illiquidity premium, we would also expect this to be reflected in a positive RAP on average. Like any other performance metric in the PE literature, our approach does not allow us to disentangle true skill from this illiquidity premium.[4] When calculating our RAP measure (and only then), we exclude vintages after 2010 for which we are still missing a substantial fraction of the cash flows.

To assess the performance of PE funds, we report both the distribution of risk-adjusted profits across all funds in the sample, as well as the equal-weighted average RAP by vintage. This approach quantifies whether the estimated replicating portfolio delivers returns that are similar to the returns realized from investing in a broad basket of PE

[3]The fund's horizon is endogenous because it is correlated with the success of the fund. As noted by Korteweg and Nagel (2016), this endogeneity does not pose a problem as long as cash-flows are observed. They write: "Even if there is an endogenous state-dependence among cash-flows, the appropriate valuation of a payoff in a certain state is still the product of the state's probability and the SDF in that state."

[4]To the best of our knowledge there is no hard evidence of the existence of an illiquidity premium. Anecdotal evidence suggests that many institutional investors such as pension funds value the fact that PE investments do not have to be marked-to-market. If those investors constitute the majority of PE investors, the illiquidity premium would in fact be negative.

funds directly. Similarity in returns between PE funds and our replicating portfolio does not follow mechanically from our replicating portfolio approach due to the requirement of budget feasibility. Our approach credits PE funds with out-performance to the extent they are able to deliver factor exposure at an (after-fee) expense lower than that of existing publicly traded assets.

Appendix D discusses the relationship between our approach and the PME approach of Kaplan and Schoar (2005) and the GPME approach of Korteweg and Nagel (2016). The rest of this section discusses implementation issues related to estimating equation (1).

2.2 Identifying and Estimating Cash-Flow Betas

The replicating portfolio must be rich enough that it spans all priced (aggregate) sources of risk, yet it must be parsimonious enough that its exposures can be estimated with sufficient precision. Allowing every fund in every category and vintage to have its own unrestricted cash-flow beta profile for each risk factor leads to parameter proliferation and lack of identification. We impose cross-equation restrictions to aid identification.

One-factor Model To fix ideas, we start with a simple model in which all private equity cash-flows are assumed to only have interest rate risk. We refer to this as the one-factor model. The empirical model assumes that the cash-flows X of all funds i in the same category c (category superscripts are omitted for ease of notation) have the same bond betas at each horizon h. To simplify the time dimension, we categorize different vintages t by the quartile of the price-dividend ratio on the stock market in the vintage quarter. Quartiles are defined over the same 1974-2017 sample for which we estimate our asset pricing model. Allowing risk exposures to scale up and down with the current value of the pd_t^m ratio captures dependence on the overall investment climate at the time of PE fund origination. The pd_t^m ratio is one of the key state variables in the asset pricing model of Section 3. The restricted cash-flow model can be expressed as:

$$X_{t+h}^{i \in c} = \beta_{t,h}^{b} + e_{t+h}^{i} = q_{t,h}^{b} + v_{t+h}^{i} = a_{pd_t^m} + b_h + v_{t+h}^{i}. \quad (7)$$

We estimate the first equation in (7) using either OLS or a lasso model. We impose that the budget-feasible bond positions q_{t+h}^{b} are the sum of a *vintage effect* a_{pd_t} and a *horizon effect* b_h. With four pd_t^m quartiles and H horizons, we estimate $H + 4$ parameters using $N_f \times T \times H$ observations, where T reflects the number of different vintages and N_f the average number of funds in a category c per vintage. Identification is achieved both from the cross-section and from the time series. Specifically, the vintage effects (the "a's") are only allowed to shift the horizon effects (the "b"-profiles) up and down in parallel fashion. The vintage effects are normalized to be zero on average across quartiles, and the horizon effects are correspondingly rescaled. We include all available vintages that have at least eight quarters of cash flows because the extra information from recent vintages may be useful to better identify the first few elements of b_h.

***K*-factor Model** Our main model is a K-factor model in which we allow for K stock market factors to proxy for PE fund cash flows. The factors are dividend strips and/or gain strips. We price these factors in the model of Section 3.

The key identifying assumption is that the cash-flows of all PE funds in the same category and whose vintage belongs to the same pd^m quartile have the same $1 \times KH$ cash-flow beta horizon profile. The beta horizon profiles on the K different factors are allowed to be different from one another, and to shift in different ways across vintages. But each pd^m quartile effect is only allowed to shift the corresponding horizon profile up and down by a fixed amount. Consider for instance an estimation on Buyout funds that consists of bonds, dividend strips on the market, and dividend strips on small stocks:

$$\begin{aligned} X^{i\in c}_{t+h} &= q^{b}_{t,h} + q^{m}_{t,h} F^{m}_{t,t+h} + q^{small}_{t,h} F^{small}_{t,t+h} + v^{i}_{t+h} \\ &= a^{1}_{t} + b^{1}_{h} + \left(a^{2}_{t} + b^{2}_{h}\right) F^{m}_{t,t+h} + \left(a^{3}_{t} + b^{3}_{h}\right) F^{small}_{t,t+h} + v^{i}_{t+h}. \end{aligned} \tag{8}$$

With $K = 3$ factors and $H = 60$ horizons, we estimate $4K = 12$ vintage-quartile effects $\{a^1_t, a^2_t, a^3_t\}_{t=1}^{4}$ and $KH = 180$ horizon profiles $\{b^1_h, b^2_h, b^3_h\}_{h=1}^{H}$ for a total of 192 coefficients.

We use two estimation techniques. The first is a standard OLS model. The second is a lasso approach that constrains all penalized coefficients to be non-negative. This second approach follows a recent literature on Machine Learning in asset pricing (e.g., Gu, Kelly, and Xiu, 2018; Kozak, Nagel, and Santosh, 2017). It aims to reduce the potential set of factors in our replicating portfolio. This estimation offers two key economic advantages. First, it constrains the replicating portfolio to long positions only, which avoids costs and difficulties related to short positions. Second, the lasso model will set to zero many possible factor and horizon terms in the replicating portfolio. This avoids having to take a stance on the identity of a small number of factors, a problem with the OLS approach. Lasso simplifies the resulting replicating positions considerably and avoids over-fitting due to extreme long-short positions. The lasso estimation of equation (1) can be written as:

$$\hat{\beta}_{lasso} = \arg \min_{\beta \in \mathbf{R}^{KH}} \| X^i_{t+h} - \beta^i_{t,h} \boldsymbol{F}_{t,t+h} \|_2^2 + \lambda_0 \mathbf{1}\{\beta > 0\} + \lambda_1 \|\beta\|_1 \tag{9}$$

We set the hyper-parameter $\lambda = \infty$, which ensures only positive coefficients, and tune the parameter λ_1 depending on the fund category.

3 Asset Pricing Model

The second main step is to price the replicating portfolio. If the only sources of risk were fluctuations in the term structure of interest rates, this step would be straightforward. After all, we can infer the prices of zero-coupon bonds of all maturities from the observed yield curve at each date. However, fluctuations in interest rates are not the only and not even the main source of risk in the cash-flows of private equity funds as we will show. If fluctuations in the aggregate stock market were the only other source of aggregate risk, then we could use price information from dividend strips. Those prices can either be observed directly from dividend strip futures markets (van Binsbergen, Hueskes, Koijen, and Vrugt, 2013) or inferred from options and stock markets (van Binsbergen, Brandt, and Koijen, 2012). However, the available time series is too short for our purposes, strips are not available for horizons beyond seven years and do not come in one-quarter horizon increments, and the only dividend strip data are for the aggregate stock market. There are no strip data for the additional traded factors we wish to include in our analysis such as publicly listed real estate or infrastructure assets, a small stock index, or a growth stock index. Finally, we do not observe expected returns on the available strips, only realized returns. For all these reasons, we use an asset pricing model to generate the time series of strip prices, $\boldsymbol{P}_{t,h}$, and the corresponding expected returns for each factor. But we impose that the asset pricing model is consistent with the available dividend strip data in addition to the standard asset pricing moments.

We propose a reduced-form stochastic discount factor (SDF) model rather than a structural asset pricing model, since it is more important for our purposes to price the replicating portfolio of publicly traded assets correctly than to understand the fundamental sources of risk that underly the pricing of stocks and bonds. Our approach builds on Lustig, Van Nieuwerburgh, and Verdelhan (2013), who price a claim to aggregate consumption and study the properties of the price-dividend ratio of this claim, the wealth-consumption ratio. A virtue of the reduced-form model is that it can accommodate a substantial number of risk factors. We argue that it is important to go beyond the aggregate stock and bond markets to capture the risk embedded in PE fund cash flows.

As in Korteweg and Nagel (2016), the objective here is not to test the asset pricing model but rather to investigate whether a potential PE investment adds value to an investor who already has access to securities whose sources of risk are captured by the SDF.

3.1 Setup

3.1.1 State Variable Dynamics

Time is denoted in quarters. We assume that the $N \times 1$ vector of state variables follows a Gaussian first-order VAR:

$$\boldsymbol{z}_t = \boldsymbol{\Psi} \boldsymbol{z}_{t-1} + \boldsymbol{\Sigma}^{\frac{1}{2}} \boldsymbol{\varepsilon}_t, \tag{10}$$

with shocks $\boldsymbol{\varepsilon}_t \sim i.i.d. \mathcal{N}(0, I)$ whose variance is the identity matrix. The companion matrix Ψ is a $N \times N$ matrix. The vector z is demeaned. The covariance matrix of the innovations to the state variables is Σ; the model is homoscedastic. We use a Cholesky decomposition of the covariance matrix, $\Sigma = \Sigma^{\frac{1}{2}} \Sigma^{\frac{1}{2}\prime}$, which has non-zero elements only on and below the diagonal. The Cholesky decomposition of the residual covariance matrix allows us to interpret the shock to each state variable as the shock that is orthogonal to the shocks of all state variables that precede it in the VAR. We discuss the elements of the state vector and their ordering below. For now, we note that the (demeaned) one-month bond nominal yield is one of the elements of the state vector: $y^{\$}_{t,1} = y^{\$}_{0,1} + e'_{yn} z_t$, where $y^{\$}_{0,1}$ is the unconditional average 1-quarter nominal bond yield and e_{yn} is a vector that selects the element of the state vector corresponding to the one-quarter yield. Similarly, the (demeaned) inflation rate is part of the state vector: $\pi_t = \pi_0 + e'_{\pi} z_t$ is the (log) inflation rate between $t-1$ and t. Lowercase letters denote logs.

3.1.2 Stochastic Discount Factor

We specify an exponentially affine SDF, similar in spirit to the no-arbitrage term structure literature (Ang and Piazzesi, 2003). The nominal SDF $M^{\$}_{t+1} = \exp(m^{\$}_{t+1})$ is conditionally log-normal:

$$m^{\$}_{t+1} = -y^{\$}_{t,1} - \frac{1}{2} \boldsymbol{\Lambda}'_t \boldsymbol{\Lambda}_t - \boldsymbol{\Lambda}'_t \boldsymbol{\varepsilon}_{t+1}. \tag{11}$$

Note that $y^{\$}_{t,1} = -\mathbb{E}_t[m^{\$}_{t+1}] - 0.5\mathbb{V}_t[m^{\$}_{t+1}]$. The real log SDF $m_{t+1} = m^{\$}_{t+1} + \pi_{t+1}$ is also conditionally Gaussian. The innovations in the vector ε_{t+1} are associated with a $N \times 1$ market price of risk vector $\mathbf{\Lambda}_t$ of the affine form:

$$\mathbf{\Lambda}_t = \mathbf{\Lambda}_0 + \mathbf{\Lambda}_1 z_t.$$

The $N \times 1$ vector $\mathbf{\Lambda}_0$ collects the average prices of risk while the $N \times N$ matrix $\mathbf{\Lambda}_1$ governs the time variation in risk premia. Asset pricing amounts to estimating the market prices of risk $(\mathbf{\Lambda}_0, \mathbf{\Lambda}_1)$. We specify the moment conditions to identify the market prices of risk below.

3.1.3 Bond Pricing

Proposition 1 in Appendix A shows that nominal bond yields of maturity τ are affine in the state variables:

$$y^{\$}_{t,\tau} = -\frac{1}{\tau}A^{\$}_{\tau} - \frac{1}{\tau}\left(B^{\$}_{\tau}\right)' z_t.$$

The scalar $A^{\$}(\tau)$ and the vector $B^{\$}_{\tau}$ follow ordinary difference equations that depend on the properties of the state vector and on the market prices of risk. The appendix also calculates the real term structure of interest rates, the real bond risk premium, and the inflation risk premium on bonds of various maturities. We will price a large cross-section of nominal bonds that differ by maturity, paying special attention to matching the time series of one- and twenty-quarter bond yields since those bond yields are part of the state vector z_t.

3.1.4 Equity Pricing

The present-value relationship says that the price of a stock equals the present-discounted value of its future cash-flows. By value-additivity, the price of the stock, P^m_t, is the sum of the prices to each of its future cash-flows D^m_t. These future cash-flow claims are the so-called dividend strips or zero-coupon equity (Wachter, 2005). Dividing by the current dividend D^m_t:

$$\frac{P^m_t}{D^m_t} = \sum_{\tau=1}^{\infty} P^d_{t,\tau} \tag{12}$$

$$\exp\left(\overline{pd} + e'_{pd^m} z_t\right) = \sum_{\tau=0}^{\infty} \exp\left(A^m_{\tau} + B^{m\prime}_{\tau} z_t\right), \tag{13}$$

where $P^d_{t,\tau}$ denotes the price of a τ-period dividend strip divided by the current dividend. Proposition 2 in Appendix A shows that the log price-dividend ratio on each dividend strip, $p^d_{t,\tau} = \log\left(P^d_{t,\tau}\right)$, is affine in the state vector and provides recursions for the coefficients $(A^m_\tau, \boldsymbol{B}^m_\tau)$. Since the log price-dividend ratio on the stock market is an element of the state vector, it is affine in the state vector by assumption. Equation (13) restates the present-value relationship from equation (12). It articulates a non-linear restriction on the coefficients $\{(A^m_\tau, B^m_\tau)\}^\infty_{\tau=1}$ at each date (for each state z_t), which we impose in the estimation. Analogous present value restrictions are imposed for the other traded equity factors, whose price-dividend ratios and dividend growth rates are also included in the state vector.

If dividend growth were unpredictable and its innovations carried a zero risk price, then dividend strips would be priced like real zero-coupon bonds. The strips' dividend-price ratios would equal yields on real bonds with the coupon adjusted for deterministic dividend growth. All variation in the price-dividend ratio would reflect variation in the real yield curve. In reality, the dynamics of real bond yields only account for a small fraction of the variation in the price-dividend ratio, implying large prices of risk associated with shocks to dividend growth that are orthogonal to shocks to bond yields.

3.1.5 Dividend Futures

The model readily implied the price of a futures contract that received the single realized nominal dividend at some future date, $D^\$_{t+k}$. That futures price, $F^d_{t,\tau}$, scaled by the current nominal dividend $D^\$_t$, is:

$$\frac{F^d_{t,\tau}}{D^\$_t} = P^d_{t,\tau} \exp\left(\tau y^\$_{t,\tau}\right),$$

The one-period realized return on this futures contract for $k > 1$ is:

$$R^{fut,d}_{t+1,\tau} = \frac{F^d_{t+1,\tau-1}}{F^d_{t,\tau}} - 1.$$

Appendix A shows that $\log(1 + R^{fut,d}_{t+1,\tau})$ is affine in the state vector z_t and in the shocks ε_{t+1}. It is straightforward to compute average realized returns over any subsample, and for any portfolio of futures contracts.

3.2 Estimation

3.2.1 State Vector Elements

The state vector contains the following 16 variables, in order of appearance: (1) GDP price inflation, (2) real per capita GDP growth, (3) the nominal short rate (3-month nominal Treasury bill rate), (4) the spread between the yield on a five-year Treasury note and a three-month Treasury bill, (5) the log price-dividend ratio on the CRSP value-weighted stock market, (6) the log real dividend growth rate on the CRSP stock market, (7) the log price-dividend ratio on the REIT index of publicly listed real estate companies, (8) the corresponding log real dividend growth rate on REITs, (9) the log price-dividend ratio on a listed infrastructure index, and (10) the corresponding log real dividend growth rate, (11) the log price-dividend ratio on the first size quintile of stocks, (12) the corresponding log real dividend growth rate, (13) the log-price dividend ratio on the first book-to-market quintile of stocks, and (14) the corresponding log real dividend growth rate, (15) the log-price dividend ratio on natural resource stocks, and (16) the corresponding log real dividend growth rate:[5]

$$
\begin{aligned}
z_t = \quad & \Big[\pi_t, x_t, y^{\$}_{t,1}, y^{\$}_{t,20} - y^{\$}_{t,1}, pd^m_t, \Delta d^m_t, pd^{reit}_t, \Delta d^{reit}_t, pd^{infra}_t, \Delta d^{infra}_t, \\
& pd^{small}_t, \Delta d^{small}_t, pd^{growth}_t, \Delta d^{growth}_t, pd^{nr}_t, \Delta d^{nr}_t\Big]'.
\end{aligned} \tag{14}
$$

This state vector is observed at quarterly frequency from 1974.Q1 until 2017.Q4 (176 observations). This is the longest available time series for which all variables are available. Our PE cash flow data starts shortly thereafter in the early 1980s. While most of our PE fund data are after 1990, we deem it advantageous to use the longest possible sample to more reliably estimate the VAR dynamics and especially the market prices of risk. All state variables are demeaned with the observed full-sample mean.

The VAR is estimated by OLS in the first stage of the estimation. We recursively zero out all elements of the companion matrix Ψ whose t-statistic is below 1.96. Appendix B.1 contains the resulting point estimates of Ψ and $\Sigma^{\frac{1}{2}}$.

[5]The model is quarterly. We use the average of daily Constant Maturity Treasury yields within the quarter. The REIT index is the NAREIT All Equity index, which excludes mortgage REITs. The first observation for REIT dividend growth is in 1974.Q1. All dividend series are deseasonalized by summing dividends across the current month and past 11 months. This means we lose the first 8 quarters of data in 1972 and 1973 when computing dividend growth rates. The infrastructure stock index is measured as the value-weighted average of the eight relevant Fama-French industries (Aero, Ships, Mines, Coal, Oil, Util, Telcm, Trans). The natural resource index is measured from the Alerian Master Limited Partnership from 1996.Q1 onwards and as the Oil industry index beforehand.

3.2.2 Market Prices of Risk

The state vector contains both priced sources of risk as well as predictors of bond and stock returns. We estimated 11 non-zero parameters in the constant market price of risk vector Λ_0 and 82 non-zero elements of the matrix Λ_1 which governs the dynamics of the risk prices. The point estimates are listed in Appendix B.2. We use the following target moments to estimate the market price of risk parameters.

First, we match the time-series of nominal bond yields for maturities of one quarter, one year, two years, five years, ten years, twenty years, and thirty years. They constitute about $7 \times T$ moments, where $T = 176$ quarters.[6]

Second, we impose restrictions that we exactly match the average five-year bond yield and its dynamics. This delivers 17 additional restrictions:

$$-A^{\$}_{20}/20 = y^{\$}_{0,20} \quad \text{and} \qquad -B^{\$}_{20}/20 = [0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0]$$

Because the five-year bond yield is the sum of the third and fourth element in the state vector, the market prices of risk must be such that $-B^{\$}_{20}/20$ has a one in the third and fourth place and zeroes everywhere else.

Third, we match the time-series of log price-dividend ratios on the stock market, real estate stocks, infrastructure stocks, small stocks, growth stocks, and natural resource stocks. The model-implied price-dividend ratios are built up from 3,500 quarterly dividend strips according to equation (12). We impose these present-value relationships in each quarter, delivering $6 \times T$ moments.

Fourth, we impose that the time series of risk premia for the six stock indices in the model match the risk premia implied by the VAR, i.e., from the data. As usual, the expected excess return in logs (including a Jensen adjustment) must equal minus the conditional covariance between the log SDF and the log return. For example, for the overall stock market:

$$\begin{aligned} E_t\left[r^{m,\$}_{t+1}\right] - y^{\$}_{t,1} + \frac{1}{2}V_t\left[r^{m,\$}_{t+1}\right] &= -Cov_t\left[m^{\$}_{t+1}, r^{m,\$}_{t+1}\right] \\ r^m_0 + \pi_0 - y^{\$}_0(1) + \left[(e_{divm} + \kappa^m_1 e_{pd} + e_\pi)'\Psi - e'_{pd} - e'_{yn}\right] z_t & \\ + \frac{1}{2}\left(e_{divm} + \kappa^m_1 e_{pd} + e_\pi\right)'\Sigma\left(e_{divm} + \kappa^m_1 e_{pd} + e_\pi\right) &= \left(e_{divm} + \kappa^m_1 e_{pd} + e_\pi\right)'\Sigma^{\frac{1}{2}}\Lambda_t \end{aligned}$$

[6] The 20-year bond yield is missing prior to 1993.Q4 while the 30-year bond yield data is missing from 2002.Q1-2005.Q4. In total 107 observations are missing, so that we have 1232-107=1125 bond yields to match.

The left-hand side is given by the VAR (data), while the right-hand side is determined by the market prices of risk Λ_0 and Λ_1 (model). This provides 75 additional restrictions. These moments help identify the 6th, 8th, 10th, 12th, and 14th elements of Λ_0 and rows of Λ_1 alongside the present-value relationships.

Fifth, we price a claim that pays the next eight quarters of realized nominal dividends on the aggregate stock market. The value of this claim is the sum of the prices to the nearest eight dividend strips. Data for the price-dividend ratio on this claim and the share it represents in the overall stock market (S&P500) for the period 1996.Q1-2009.Q3 (55 quarters) are obtained from van Binsbergen, Brandt, and Koijen (2012). This delivers 2×55 moments. We also ensure that the model is consistent with the high average realized returns on short-horizon dividend futures, first documented by van Binsbergen, Hueskes, Koijen, and Vrugt (2013). Table 1 in van Binsbergen and Koijen (2017) reports the observed average monthly return on one- through seven-year U.S. SPX dividend futures over the period Nov 2002 - Jul 2014. That average portfolio return is 8.71% per year. We construct an average return for the same short maturity futures portfolio (paying dividends 2 to 29 quarters from now) in the model:

$$R_{t+1}^{fut,portf} = \frac{1}{28} \sum_{\tau=2}^{29} R_{t+1,\tau}^{fut,d}$$

We evaluate the realized return on this dividend futures portfolio at the state variables observed between 2003.Q1 and 2014.Q2, average it, and annualize it. This results in one additional restriction. We free up the market price of risk associated with the market price-dividend ratio (fifth element of Λ_0 and first six elements of the fifth row of Λ_1) to help match the dividend strip evidence.

Sixth, we impose a good deal bound on the standard deviation of the log SDF, the maximum Sharpe ratio, in the spirit of Cochrane and Saa-Requejo (2000).

Seventh, we impose regularity conditions on bond yields. We impose that very long-term real bond yields have average yields that weakly exceed average long-run real GDP growth, which is 1.65% per year in our sample. Long-run nominal yields must exceed long-run real yields by 2%, an estimate of long-run average inflation. These regularity conditions are satisfied at the final solution.

Not counting the regularity conditions, we have 2,385 moments to estimate 93 market price of risk parameters. The estimation is massively over-identified.

3.2.3 Model Fit

Figure 1 plots the bond yields on bonds of maturities 1 quarter, 1 year, 5 years, and 10 years. Those are the most relevant horizons for the private equity cash-flows. The model matches the time series of bond yields in the data closely for the horizons that matter for PE funds (below 15 years). It matches nearly perfectly the 1-quarter and 5-year bond yield which are part of the state space.

The top panels of Figure 2 show the model's implications for the average nominal (left panel) and real (right panel) yield curves at longer maturities. These long-term yields are well behaved. The bottom left panel shows that the model matches the dynamics of the nominal bond risk premium, defined as the expected excess return on five-year nominal bonds. The compensation for interest rate risk varies substantially over time, both in data and in the model. The bottom right panel shows a decomposition of the yield on a five-year nominal bond into the five-year real bond yield, annual expected inflation over the next five years, and the five-year inflation risk premium. On average, the 5.7% five-year nominal bond yield is comprised of a 1.7% real yield, a 3.3% expected inflation rate, and a 0.8% inflation risk premium. The importance of these components fluctuates over time.

Figure 3 shows the equity risk premium, the expected excess return, in the left panels and the price-dividend ratio in the right panels. The top row is for the overall stock market, the second row for REITs, the third row for infrastructure stocks, fourth row for small stocks, the fifth row for growth stocks, and the last row for natural resource stocks. The dynamics of the risk premia in the data are dictated by the VAR. The model chooses the market prices of risk to fit these risk premium dynamics as closely as possible.[7] The price-dividend ratios in the model are formed from the price-dividend ratios on the strips of maturities ranging from 1 to 3500 quarters, as explained above. The figure shows an excellent fit for price-dividend levels and a good fit for risk premium dynamics. Some of the VAR-implied risk premia have outliers which the model does not fully capture. This is in part because the good deal bounds restrict the SDF from becoming too volatile and extreme. We note large level differences in valuation ratios across the various stock factors, as well as big differences in the dynamics of risk premia and price levels.

[7] The quarterly risk premia are annualized (multiplied by 4) for presentational purposes only. The VAR does not restrict risk premia to remain positive. The VAR-implied market equity risk premium is negative in 21% of the quarters. For REITs this is 10% and for infrastructure only 5% of quarters. The most negative value of the risk premium on the market is -8% quarterly. For REITs the most negative value for the risk premium is -2.9% quarterly, while it is -1.2% for infrastructure.

3.3 Temporal Pricing of Risk

The first key input from the model into the private equity valuation exercise are the prices of nominal zero-coupon bonds and of the various dividend strips; recall equation (2). Figure 4 plots these strip prices, scaled by the current quarter dividend. For readability, we plot only three maturities: one, five, and ten years. The model implies substantial variation in strip prices over time, across maturities, as well as across risky assets. If the replicating portfolio for VC funds originated in the year 2000 loads heavily on growth strips, when growth strips are expensive, then all strip positions need to be scaled down in order to make the replicating portfolio budget feasible. This increases the risk-adjusted performance of vintage-2000 VC funds, all else equal.

As part of the estimation, the model fits several features of traded dividend strips on the aggregate stock market. Figure 5 shows the observed time series of the price-dividend ratio on a claim to the first 8 quarters of dividends (red line, left panel), as well as the share of the total stock market value that these first eight quarters of dividends represent (red line, right panel). The blue line is the model. The model generates the right level for the price-dividend ratio for the short-horizon claim. For the same 55 quarters for which the data are available, the average is 7.75 in the model and 7.65 in the data. The first 8 quarters of dividends represent 3.4% of the overall stock market value in the data and 4.5% in the model, over the period in which there are data. The model mimics the observed dynamics of the short-horizon value share quite well, including the sharp decline in 2000.Q4-2001.Q1 when the short-term strip value falls by more than the overall stock market. This reflects the market's perception that the recession would be short-lived. In contrast, the share of short-term strips increases in the Great Recession, both in the data and in the model, in recognition of the persistent nature of the crisis.

The second key input from the model into the private equity valuation exercise are the expected excess return on the bond and stock strips of horizons of 1-60 quarters. After all, the expected return of the PE-replicating portfolio is a linear combination of these expected returns; recall equation (3). Figure 6 plots the average risk premium on nominal zero coupon bond yields (top left panel) and on all dividend strips (other panels). Risk premia on nominal bonds are increasing with maturity from 0 to 3.5%. The second panel shows the risk premia on dividend strips on the overall stock market (solid blue line). It also plots the dividend futures risk premium. The difference between the dividend spot and futures risk premium is approximately equal to the nominal bond risk premium. The unconditional dividend futures risk premium (red line) is downward sloping in maturity at the short end of the curve, and then flattens out. The graph also plots the model-implied dividend futures risk premium, averaged over the period 2003.Q1-2014.Q2 (yellow line). It is substantially more downward sloping at the short end than the risk premium averaged over the entire 1974-2017 sample. Indeed, the model matches the *realized* portfolio return on dividend futures of maturities 1-7 years over the period 2003.Q1-2014.Q2, which is 8.7% in the data and 8.7% in the model.[8]

The remaining four panels of Figure 6 show the dividend strip risk spot and future premia for real estate, infrastructure, small, growth, and natural resource stocks. Average futures risk premia are generally declining to flat in maturity. The upward slope in spot risk premia is inherited from the nominal bond risk premia. Heterogeneity in risk premia by asset class, by horizon, and over time will give rise to heterogeneity in the risk premia on the PE-replicating portfolios.

Appendix C provides further insight into how the model prices risk at each horizon using the shock exposure and shock price elasticity tools developed by Hansen and Scheinkman (2009) and Borovička and Hansen (2014).

[8] As an aside, the conditional risk premium, which is the *expected* return on the dividend futures portfolio over the 2003.Q1-2014.Q2 period is 6.0% per year in the model. The unconditional risk premium on the dividend futures portfolio (over the full sample) is 5.2%.

4 Expected Returns and Risk-adjusted Profits in PE Funds

In this section, we combine the cash-flow exposures from section 2 with the asset prices from section 3 to obtain risk-adjusted profits on private equity funds.

4.1 Summary Statistics

Our fund data cover the period January 1981 until December 2017. The data source is Preqin. We group private equity funds into seven categories: Buyout (LBO), Venture Capital (VC), Real Estate (RE), Infrastructure (IN), Fund of Funds (FF), Debt Funds (DF), and Restructuring (RS). Our FF category contains the Preqin categories Fund of Funds, Hybrid Equity, and Secondaries. The Buyout category is commonly referred to as Private Equity, whereas we use the PE label to refer to the combination of all investment categories.[9]

We include all funds with non-missing cash-flow information. We group funds also by their vintage, the quarter in which they make their first capital call. The last vintage we consider in the analysis is the 2017.Q4 vintage. Table 1 reports the number of funds and the aggregate AUM in each vintage-category pair. In total, we have 4,219 funds in our analysis and an aggregate of $4.1 trillion in assets under management. There is clear business cycle variation in when funds funds get started as well as in their size (AUM). Buyouts are the largest category by AUM, followed by Real Estate, and then Venture Capital. The last column of the table shows the quartile of the price/dividend ratio on the aggregate market, which we use as a conditioning variable, averaged over the quarters (vintages) within each year.

Figure 7 shows the average cash-flow profile in each category for distribution events, pooling all funds and vintages together and equally weighting them. We combine all monthly cash-flows into one yearly cash-flow for each fund. The first bar corresponds to the first year of the fund's vintage, while the last bar includes the discounted sum of cash flows that occur after the fifteenth year of fund origination under a separately highlighted color (green). We similarly collapse all post-15 year cash flows to the last quarter in our estimation. The literature has commonly seen private equity vehicles as lasting for about ten years, and we do observe that the majority of distribution cash-flows occur between years 5 and 10. However, we observe that late-in-life and terminal cash flows account for a substantial portion of the total cash received by LPs (especially for IN and VC funds)

[9]One may be able to further enrich the analysis by defining categories more granularly. For example, real estate strategies are often subdivided into opportunistic, value-add, core plus, and core funds. Infrastructure could be divided into greenfield and brownfield, etc.

and therefore incorporate them in our analysis. Industry publications have also noted the increasing lifespan of private equity funds.[10]

Figure 8 zooms in on the four investment categories of most interest to us: LBO, VC, RE, and IN. The figure shows the average cash-flow profile for each vintage. Since there are few LBO and VC funds prior to 1990 and few RE and IN funds prior to 2000, we start the former two panels with vintage year 1990 and the latter two panels with vintage year 2000. The figure shows that there is substantial variation in cash-flows across vintages, even within the same investment category. This variation will allow us to identify vintage effects. Appendix Figures E.1 shows cash-flow profiles for the remaining categories.

The figure also highlights that there is a lot of variation in cash-flows across calendar years. VC funds started in the mid- to late-1990s vintages realized very high average cash-flows around calendar year 2000 and a sharp drop thereafter. Since the stock market also had very high cash-flow realizations in the year 2000 and a sharp drop thereafter, this type of variation will help the model identify a high stock market beta for VC funds. This is an important distinction with other methods, such as the PME, which assume constant risk exposure and so would attribute high cash flow distributions in this period to excess returns.

4.2 Factor Estimation in OLS and Lasso

We start with a discussion of the factor loadings estimated in our model. We compare two approaches, both of which are run separately for each fund category. The first is a two-factor model (stock dividend strips and bond strips) estimated through OLS. The second is a lasso model estimated on a full set of thirteen factor (bonds strips, and dividend and capital gains strips for: aggregate stock index, small stocks, REIT stocks, growth stocks, infrastructure stocks, and natural resources).

The key parameters from both models model are the factor exposures across each horizon, corresponding to the replicating portfolio exposure in a zero-coupon bond or equity strip payoff in that same horizon. We additionally allow these factor exposures to vary depending on the price/dividend quartile in the originating vintage to account for general market conditions at the time of fund formation.

The lasso model differs from the OLS model in two ways. The first is a positivity con-

[10]For instance, a Preqin report from 2016 remarks: "The average lifespan of funds across the whole private capital industry is increasing beyond the typical 10 years... older funds of vintages 2000-2005 still hold a substantial $204bn worth of investments, equating to 7.2% of total unrealized assets" (Preqin, 2016).

straint, which constrains all $\beta_{t,h}$ coefficient estimates to be positive. The resulting replicating portfolio consist of long-only exposures. We implement this requirement by imposing a penalty on any negative coefficient estimates, and "shrink" such negative coefficients to be precisely zero. Additionally, the lasso estimation also includes a tuning parameter λ_1, which also shrinks coefficients to zero if they do not add sufficient estimated impact to the model fit.

This lasso approach has the main benefit that it results in substantial dimension reduction of our estimation problem, which is essential in estimating a large number of parameters across a variety of horizons, factors, and underlying vintage states. Absent this dimensionality reduction, we simply would be unable to estimate an asset pricing model with a rich set of possible factors given our limited number of PE funds. Despite the large number of possible factor exposures, applying penalized terms to our estimation results in a parsimonious replicating portfolio.

For both sets of models, we display scaled coefficients corresponding to replicating portfolio position weights. We scale the resulting positions in the zero coupon bonds and equity dividend and capital gains strips of various maturities based on the model prices to ensure budget feasibility. The high cash flows of a particular PE category may not be achievable/replicable with a budget-feasible bond portfolio, but only with a budget infeasible one. This will result in high "errors" v and high average risk-adjusted profits across the funds in that vintage-category.

Figure 9 contrasts our resulting factor exposures along the dimension of horizon $\widehat{b}_h$ obtained through both the 2-factor OLS model (left) and the lasso model using the full set of factors (right). Each row corresponds to a PE category. Appendix Figure E.2 contains these estimates for the other fund categories. Appendix Figures E.3 and E.4 contain these estimates for the PD-quartile effects $\widehat{a}_t$.

Our OLS estimation builds on previous work such as Korteweg and Nagel (2016) in estimating PE fund exposure to equity and bond factors. Our approach differs, however, in several respects. First, we use bond and stock strips, rather than realized equity and bond returns to match PE cash flows. This dividend strip approach allows for considerable flexibility in estimating stock and equity exposures that vary potentially each year in the horizon of the fund, corresponding to cash flows that are more bond or equity-like throughout the fund lifecycle. Additionally the fund exposures can vary in the time-series according to the valuation ratio at the time of fund inception.

For Buyout funds in Panel A, we find substantial bond exposures throughout most of the life of the fund. These are typically matched with short positions on bonds, with the exception of the terminal cash flow represented by year 16. This reflects all future cash flows distributed by the fund beyond year fifteen, discounted to year 15. This terminal cash exposure better fits a bond than equity position.

The lasso model on the right, by contrast, differs in several respects. First, all positions are positive, reflecting the role of the positivity constraint. This ensures that portfolios are replicable through long-only positions. Second, as previously discussed we introduce a much larger set of factors than is typically considered in the literature. Third, we impose a lasso constraint on the coefficient values that varies by category.[11] This ensures sufficient dimension reduction among the set of our factors in order to produce a sparse portfolio of replicating positions in order to match PE fund cash flows. We see that the two factors we chose in the OLS model for Buyout funds find little weight in the lasso model, suggesting that other cross-sectional factors result in better model fit.

The most salient of these factors are three gains strips: stock gain (both early in fund life and for the terminal cash flow), small gain (reaching a maximum around year five), and growth gains (greatest later in fund life).. This replicating strategy approximates the cash flows generated by exits from PE fund investments, which also result in capital gain exposure.

In addition to these gains strips, we also find some moderate evidence for dividend cash flows (predominately coming from infrastructure dividend strips) and some from other categories (such as REIT gains strips, natural resource dividend and gains strips) in ways which vary over the fund lifecycle.

While the resulting pattern of dividend strip exposure to PE funds is rich and complex, we emphasize our model picks up exposures which are economically sensible. Our replicating portfolio places strong emphasis on cross-sectional equity factors, particularly on the gains strips, that correspond to Buyout fund activities which consist of purchasing companies with the intent to restructure and resell. The key contribution of our paper lies in a methodology which is able to more accurately assess the cross-sectional factor loading of these private vehicles. A central finding which results from this approach is the rich dynamics in the estimated factor exposures across fund life and in the time-series which suggests that the analysis of risk and return in PE should take into account a more complex factor structure than has traditionally been explored.

We see further evidence of this cross-sectional factor loading in Panel B, which exam-

[11] For Buyout funds, we set $\lambda_1 = 5$; for Venture Capital we set $\lambda_1 = 1$, for Real Estate we set $\lambda_1 = 0.5$; and for the remaining categories we set $\lambda_1 = 0.25$.

ines Venture Capital funds. Our OLS model places some weight on the equity dividend factor both early and late in fund life, but finds that a bond loading best fits cash distributions for the most active middle of the fund life. Instead, our lasso model indicates that the single factor which contributes most to the fit of VC cash flows for the first nine years of fund life is growth gains strips; selling firms in the bottom quintile of the book-to-market distribution. Appendix Figure E.3 suggests that this loading is even higher in periods when the market has a higher price/dividend ratio.

Our findings for VC funds carry an important economic intuition. While Buyout funds acquire a range of companies which may differ in their underlying factor exposures; VC funds concentrate on early-stage and rapidly expanding entrepreneurial companies and distribute little cash prior to their exits from these funds. Correspondingly, we find that the bulk of VC fund cash flow exposure can be accounted for by growth gains strips, However, we do find evidence for considerable idiosyncratic volatility in fund exposures in our baseline estimation, which is run as a panel regression including all funds. We discuss later other portfolio-based strategies intended to better assess model fit.

Panel C repeats our estimation for Real Estate funds. Here, we find that stock dividend strips tend to have a positive loading throughout fund horizon in the OLS model. However, our lasso model picks up additional cross-sectional exposures which crowd out this stock dividend exposure. Not surprisingly, given the category, REIT dividends and REIT gains strips are important components of the overall fit. Infrastructure dividends strips, growth gains, stock gains, and small stock gains are additional components which appear in the estimation. These results suggest that Real Estate funds take on a distinct factor exposure profile which does incorporate some standard factors which tend to be found in most categories; but additionally incorporate factors suggesting a sector-specific profile.

Panel D continues our analysis on Infrastructure funds. Here, too, we find a strong role for some sector-specific factors, such as infrastructure dividend strips, natural resource gains strips, and REIT gains strips which point to the role of underlying asset characteristics in driving the fund-level asset pricing profile. Interestingly, the infrastructure category tends to place greater weight on dividend strips, as opposed to capital gains strips; suggesting that the cash flows in this sector are more like dividends than like realized prices.

These rich dynamics across horizon, price-to-dividend ratios, and choice of factors have important asset pricing implications. A key takeaway is that the risk loadings on

PE funds broadly cannot be assumed to be static either in the time-series or across the maturity of fund age. Additionally, they exhibit important sector-specific variation. A generic finding is that the sector-specific factors frequently offer better predictive fit for each of the categories (aggregate stock, small, and growth for PE; growth for VC; REIT for real estate; and listed infrastructure stocks for infrastructure funds). We provide the first systematic analysis of the asset pricing properties of these some of these alternative fund categories, and find that they carry this sector-specific asset exposure. These exposures are generally not constant across fund life; but are frequently concentrated in the first half of the fund cash flows.

As a consequence, the analysis of risk and return for these sorts of funds must take into account a more accurate assessment of the relevant factor risk for the different categories. Our dividend-strip estimation approach allows us to translate these complex risk dynamics into the expected return for different fund categories and to revisit the question of performance evaluation.

4.3 Expected Return

With the replicating portfolio of zero-coupon bonds and dividend strips in hand (details in Appendix A), we can calculate the expected return on PE funds in each investment category as in equation (3). Figure 10 plots the time-series of the expected return; by aggregating all of the different horizon effects; and annualizing the resulting expected return as in equation (5). The left panels of this figure illustrate the 2-factor OLS model; the right panels focus on the lasso model with 13 factors.[12]

We also observe interesting patterns in the time-series of expected returns. Time variation in the factor exposure, through dependence on the pd-quartile of the vintage, but especially through the state variables of the VAR drive variation in the expected return of the replicating portfolio of the PE fund. Our results suggest that the annualized expected returns that investors can anticipate in their PE investments has seen large variation over time, with a declining pattern at low frequencies.

In the OLS models, on the left side, we observe dramatically lower expected returns in the post-2000 sample. In turn, this implies that PE investors should expect to see lower returns in the later periods. We also observe high-frequency spikes in the expected return, corresponding to periods of low price-dividend ratios which are hit especially in 2009. In the RE category, we observe substantially higher expected returns in the 1980s (peaking

[12] Appendix Figure E.5 highlights these results for other categories.

at 40%), and a large spike in 2009 as well. The IN category sees much lower expected returns in the early part of the sample, by contrast, before peaking at 25% in the period from 1985–1995.

The lasso model, by comparison, sees generally less extreme estimates. We observe lower expected returns after 2000, but not to the same degree. For Buyout and VC funds in this lasso model, we observe expected returns reaching a maximum of about 25% per year in the 1980s, and declining to close to zero around the 200s. More recent vintages see expected returns in the 10–15% range, corresponding to typical rates of return anticipated by LPs.

Our lasso estimates, in particular, generate realistic expected returns for PE categories. These expected returns are driven by our estimates of cross-sectional factor exposure; factor exposure in the time-series varying on the price-dividend ratio; and our model implied prices for different factor exposures.

4.4 Performance Evaluation

Next, we turn to performance evaluation in the context of both the 2-factor OLS model and the lasso model. We do so by plotting directly overlapping profit histograms resulting from our two models in Figure 11.[13]

To generate an estimate of Risk-Adjusted Profit (RAP) for each fund, we compare the realized distributions against the payoffs from the replicating portfolio as in 6. We estimate the profit for each horizon, and compute the discounted sum of all profits to generate a RAP fund-by-fund. The gray histogram in this figure shows the profit distribution under the OLS 2-factor model, while the yellow histogram bars display the distribution under the lasso model with all of the factors.

In Buyout, Real Estate, and Infrastructure: we observe that the entire profit distribution is shifted to the left under the lasso model. This suggests that our estimate of fund profits are shrunk closer to zero when we account for more of the cross-sectional factor exposure through our lasso model.

As a result of this factor exposure, an LP using traditional approaches would attribute a fund's performance to true alpha, or outperformance, which our lasso model would instead attribute in part to compensation for factor exposure.

By contrast, our estimates in the VC category suggest higher risk-adjusted outperformance when controlling for a broader range of factors. This suggests that Venture Funds

[13] Alternate categories are shown in Figure E.6.

have greater outperformance when measured relative to a series of well-matched factors corresponding to growth gains; as opposed to with just an equity factor.

In Figure 12, we also plot the RAP over time for both the OLS (left) and lasso (right) models.[14] While the general time-series path for profits are similar across boths sets of models, there are also notable differences. For instance, in the Buyout category, we observe positive profits continuing through the most recent vintages in the OLS model. By contrast, our lasso model estimates positive profits for historic vintages, but close to zero profits in the most recent vintages; suggesting that these funds on average offer little risk-adjusted outperformance.

The starkest role for risk-adjustment can be seen in the VC category. While we observe extremely high profits for funds originated in vintages from the early 1990s; these profits have come down close to zero for vintages since 2000.

4.5 Model Comparison

To better benchmark our model estimates, and compare against other approaches, in Table 2 we highlight model characteristics across a range of models. The first three reflect standard measures to evaluate PE funds: the TVPI, IRR, and PME across each fund category.

Next, we display our 2-factor lasso model including stock dividends and bonds. We show the R^2, RAP, and standard deviation of the RAP. For Buyout fund, this model has an R^2 of 0.153, and a profit of 32 for each dollar invested in the PE fund. Relative to the PME, this lower profit estimate reflects estimated factor estimate, compared with the PME which assumes that all funds have an equity beta of one.

Next, we add additional factors and estimate the impact on model fit and RAP. The three factor model consists of the following factors. For Buyout, these are: stocks, bonds, and small stocks; for VC, those are: small stocks, growth stocks, and bonds; for Real Estate they are: bonds, stocks, and REIT; and for infrastructure the categories include: stocks, bonds, and infrastructure. The 5 factor model model includes the two gains strips corresponding to the equity dividend factors in addition. Additionally we include estimates that include all of the dividend factors, and a full 13 factor model.

Across the different models, we observe steadily increasing model fit by increasing the number of models. For Buyout, we increase our model fit to 0.16 when adding a list

[14] Appendix Figure E.7 plots these estimates for alternate fund categories.

of all factors; while going from 0.041 to 0.06 in VC; 0.178 in RE to 0.2; and from 0.089 in IN to 0.1 with the full factor model.

In addition to this moderate improvement in model fit, we also observe substantially different profit estimates across different models. In comparison with the two-factor model, our estimate in the Buyout category for instance shows substantially lower profit estimates (0.266 compared with 0.32), suggesting that under the more complex model, we obtain a much lower estimate of risk-adjusted outperformance. We also observe this reduction of our profit estimates in the Real Estate category, while they are more constant in the Infrastructure category.

By VC, in contrast, we tend to observe greater profits as we increase the number of possible factors; and especially once we include capital gains strips (especially the growht gains strip) in the model. This suggests that VC funds can be considered to be moderatley outperforming in relation to a series of factors which includes standard equity strips; but they outperform to a greater extent when compared with comparable growth assets.

4.6 Comparison with Other PE Performance Approaches

We benchmark our results against other commonly used PE fund performance metrics in Figure based on the lasso model. Appendix Figure does the same for our OLS model.

The left panels of this figure plots fund-level IRR against our RAP measure The right panels plot a comparison of the Kaplan and Schoar (2005) PME measure against our measure of RAP.

The key takeaway from a comparison of various approaches is the broad similarity of performance evaluation. Our measure of RAP generally correlates between 0.7–0.9 with the IRR and PME approaches. The correlation is generally slightly higher in comparison with the K-S PME as opposed to the IRR in the Buyout and VC categories for which we have the most data. This is reasonable as the PME approach also incorporates a role for public market assets. This similarity indicates that our measure of RAP generally agrees with the other commonly accepted measures of PE performance, lending some credibility to our approach. The measures are not identical, however, so that there will be funds which conventional measures assess to be high-performing but our estimates suggest only offer fair (or too little) compensation for factor risk.

Additionally, the key ways in which our approach differs from conventional measures lies in its ability to evaluate factor loadings and expected returns across different fund categories and different horizons. These results here indicate that our approach is able to considerably deepen our analysis of the asset pricing characteristics of privately listed funds, while refining the broad conclusions about performance in prior literature.

5 Conclusion

We provide a novel valuation method for private equity cash-flows that decomposes the cash-flow at each horizon into a systematic component that reflects exposure to various sources of aggregate risk, priced in listed securities markets, and an idiosyncratic component which reflects the risk-adjusted profit to the PE investor. The systematic component represents a portfolio of stock and bond strips paying safe or risky cash flows at horizons over which PE funds make cash flow distributions. A state-of-the-art no-arbitrage asset pricing model estimates prices and expected returns for these strips, fitting the time series of bond yields and stock prices, including dividend strips.

Using both OLS and lasso approaches, we estimate rich heterogeneity in PE fund risk exposures across horizons, in the cross-section, and in the time-series. PE fund risk exposure is best modeled not only using bonds and stocks; but is improved with the addition of sector-specific factor exposures. Our estimated cross-sectional exposures are sensible given the nature of underlying assets, including growth gains strips for VC and REIT dividend and growth strips for real estate funds. In the time series, expected returns on PE investments have been declining substantially between the 1980s and 2010s.

On average, PE funds considerably outperform their replicating portfolio benchmark, suggesting that they can offer investors access to these various risk factors at a cheaper price than public markets. However, while our resulting profit measures correlate well with existing measures; they suggest differing conclusions regarding the performance of the industry. Our lasso model using all factors points to low profits in the most recent vintages for VC and Buyout funds, and considerably less profits on average than indicated by traditional approaches or more simple models.

Our analysis highlights the value of a methodological advance in the assessment of risk and return for unlisted assets, which are an increasing component of the total investable universe for many institutional investors. While Private Equity is an especially important application of our approach, given the size of this category, our method can be applied more broadly to study the asset pricing characteristics of project finance and any other cash-flowing asset that is not listed on the capital markets.

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FIGURE 1: Dynamics of the Nominal Term Structure of Interest Rates

The figure plots the observed and model-implied 1-, 4-, 20-, 40-quarter nominal bond yields.

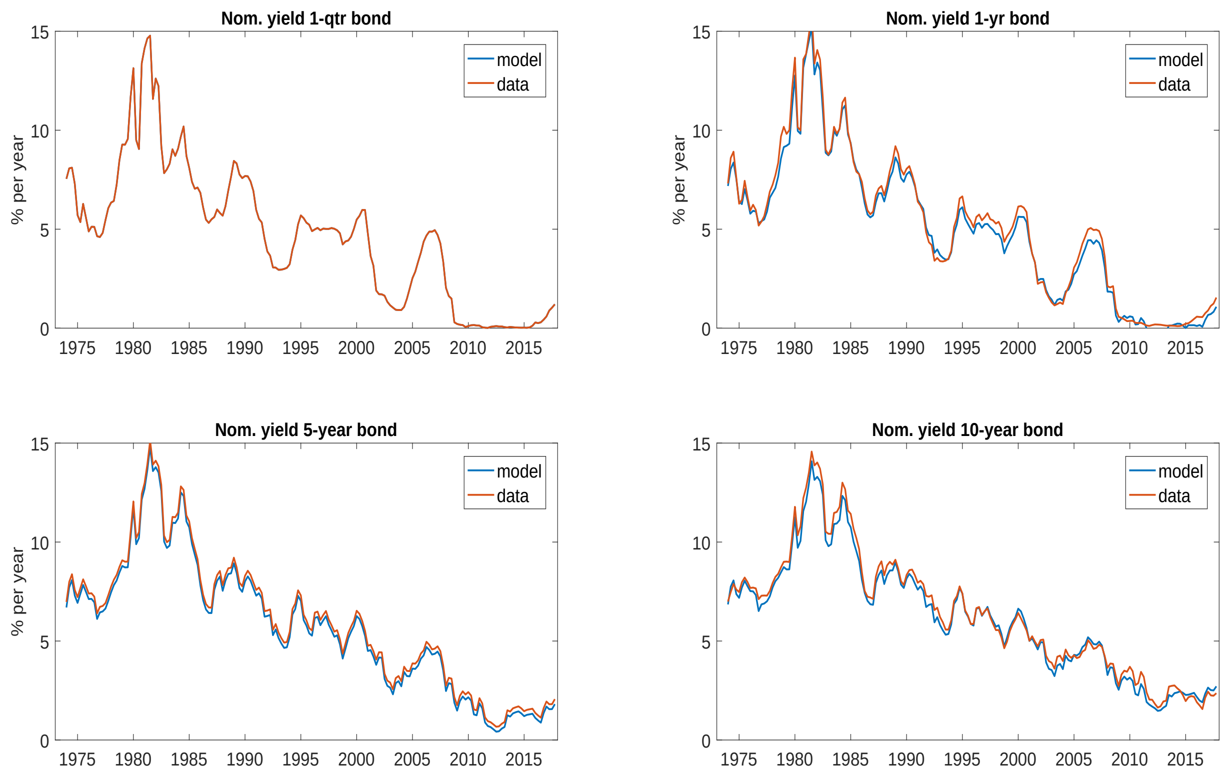


FIGURE 2: Long-term Yields and Bond Risk Premia

The top panels plot the average bond yield on nominal (left panel) and real (right panel) bonds for maturities ranging from 1 quarter to 200 quarters. The bottom left panel plots the nominal bond risk premium in model and data. The bottom right panel decomposes the model's five-year nominal bond yield into the five-year real bond yield, the five-year inflation risk premium and the five-year real risk premium.

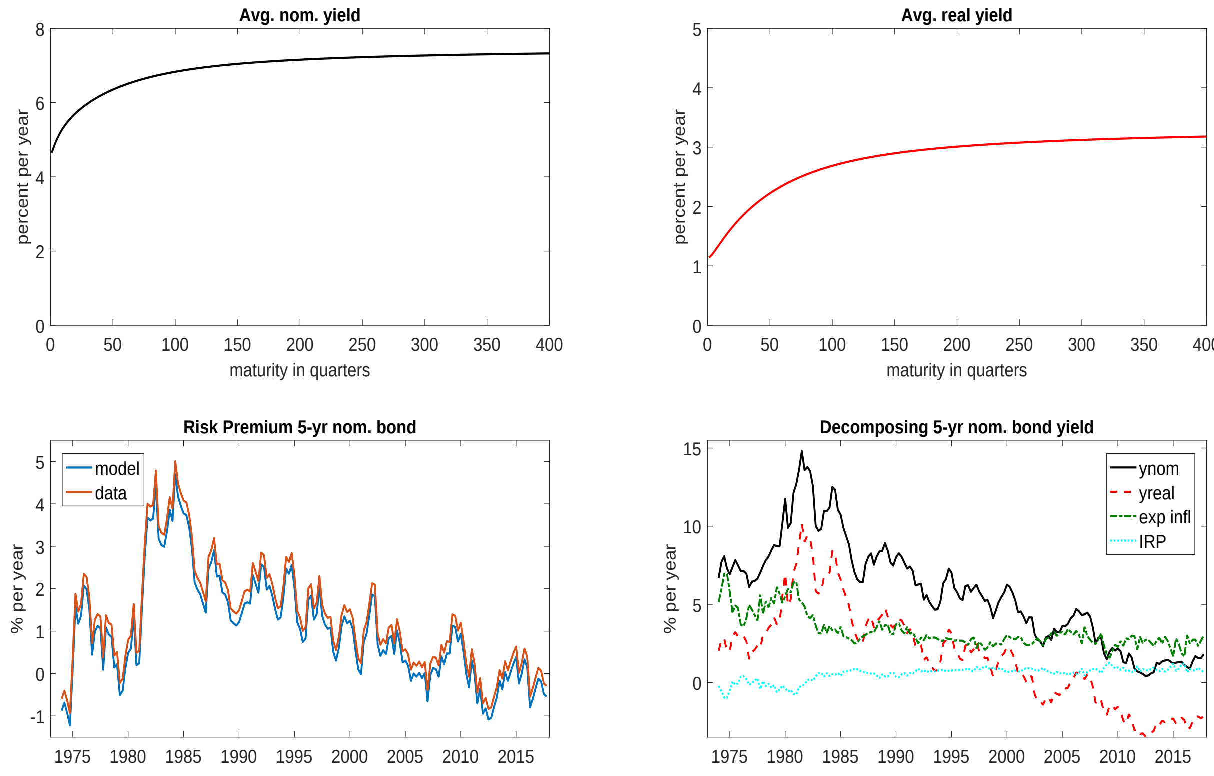


FIGURE 3: Equity Risk Premia and Price-Dividend Ratios

The figure plots the observed and model-implied equity risk premium on the overall stock market, REIT stocks, infrastructure stocks, small stocks, growth stocks, and natural resource stocks, in the left panels, as well as the corresponding price-dividend ratio in the right panels. The model is the blue line, the data are the red line.

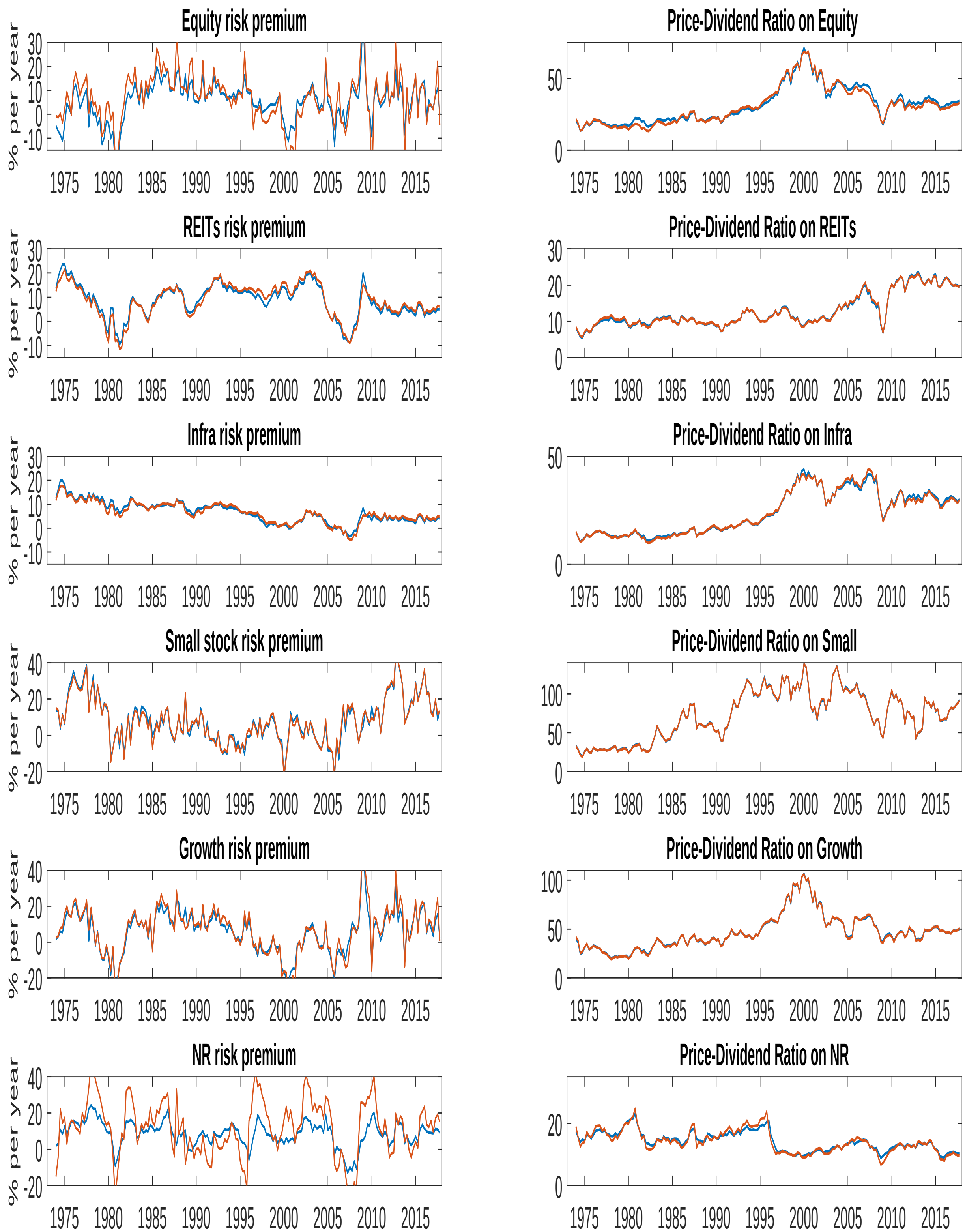


FIGURE 4: Zero Coupon Bond Prices and Dividend Strip Prices

The figure plots the model-implied prices on zero-coupon Treasury bonds in the first panel, and price-dividend ratios for dividend strips on the overall stock market, REIT market, infrastructure stocks, small stocks, and growth stocks in the next five panels, for maturities of 4, 20, and 40 quarters. The prices/price-dividend ratios are expressed in levels and each claim pays out a single cash flow.

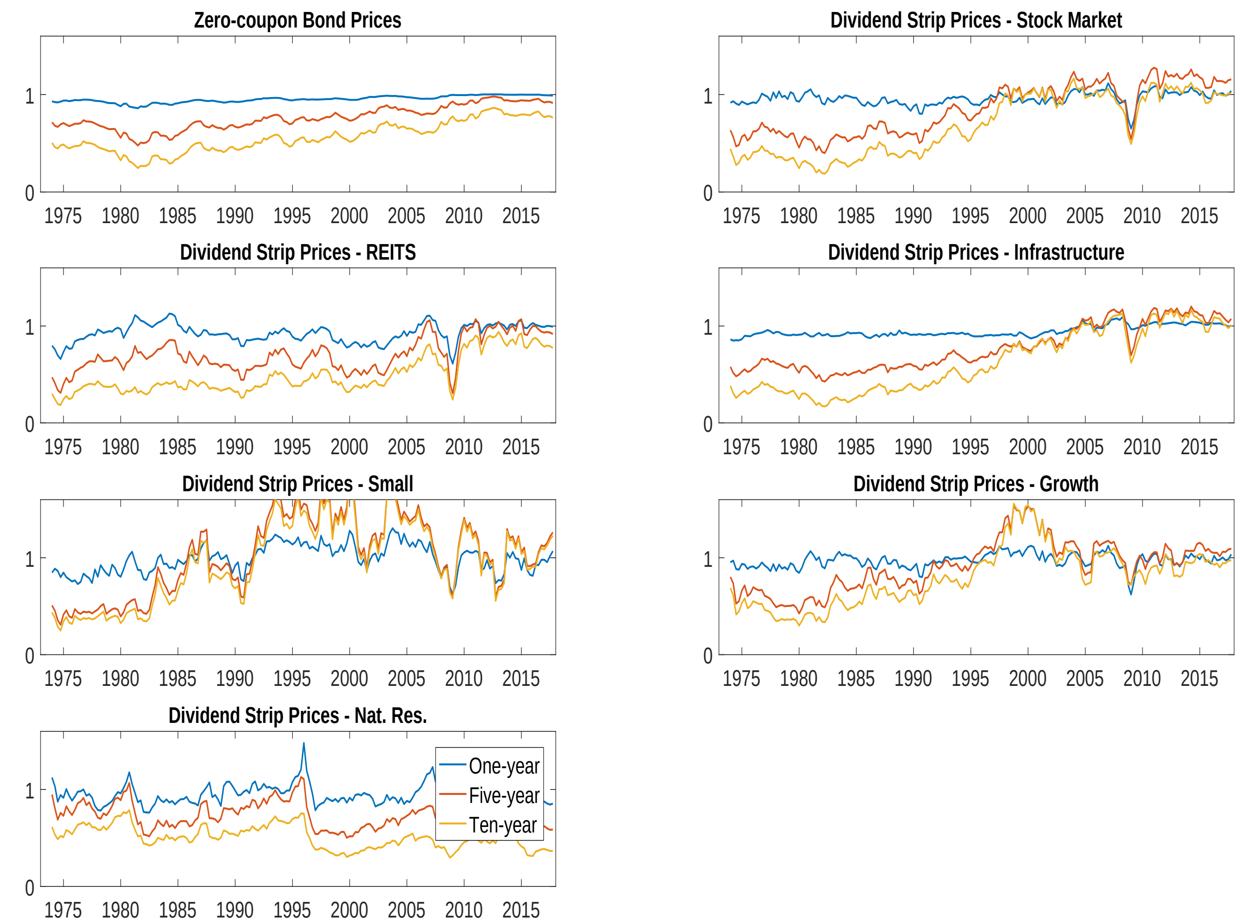


FIGURE 5: Short-run Cumulative Dividend Strips

The left panel plots the model-implied price-dividend ratio on a claim that pays the next eight quarters of dividends on the aggregate stock market. The right panel plots the share that this claim represents in the overall value of the stock market. The data are from van Binsbergen, Brandt, and Koijen (2012) and available from 1996.Q1-2009.Q3.

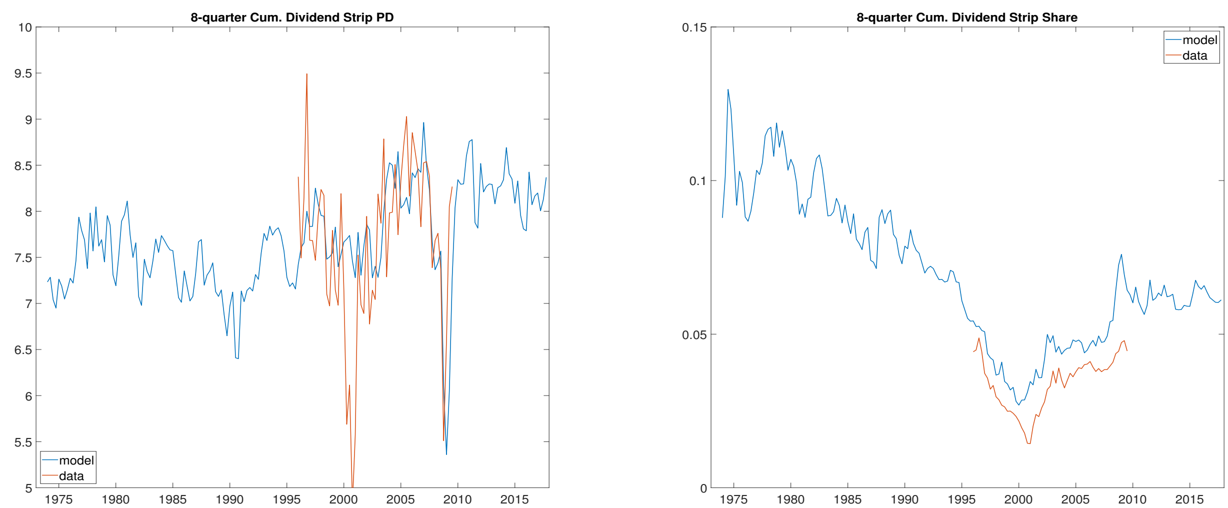


FIGURE 6: Strip Expected Returns

The figure plots the model-implied average risk premia on nominal zero-coupon Treasury bonds in the first panel, and on dividend strips on the overall stock market, REIT market, and infrastructure sector in the next three panels, for maturities ranging from 1 to 180 months.

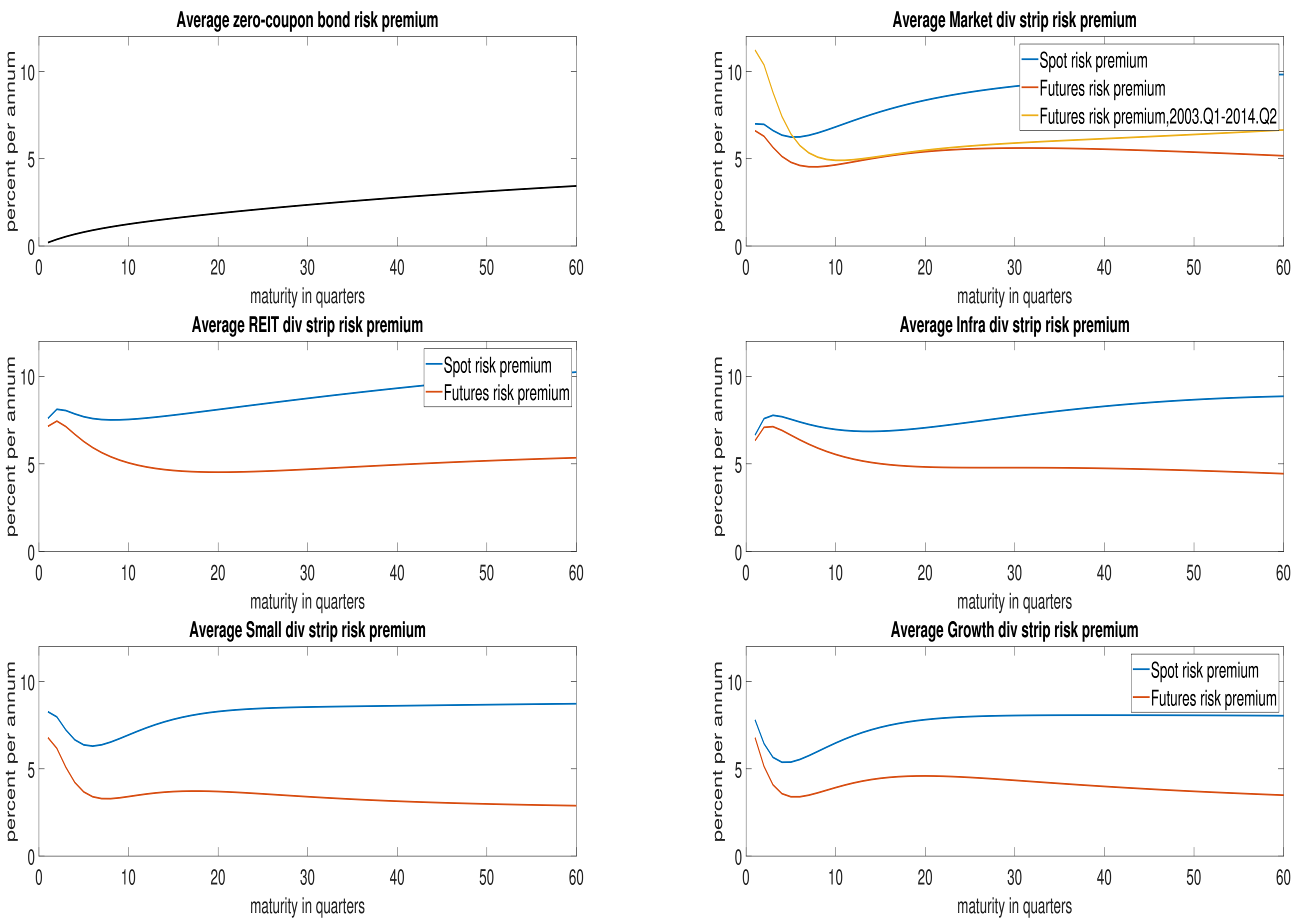


FIGURE 7: Distribution Cash-flow Profiles

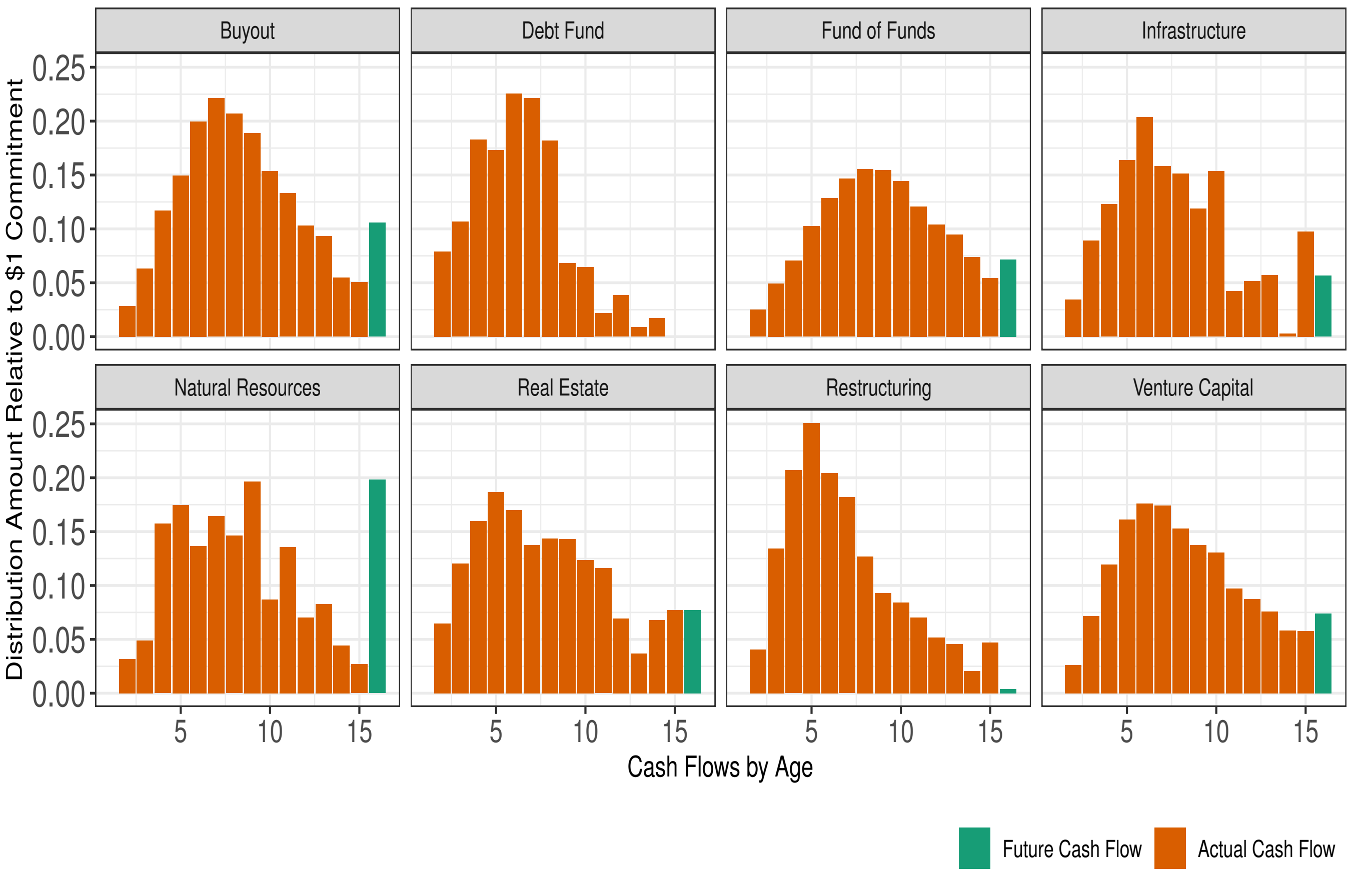


FIGURE 8: Cash-flows by Vintage

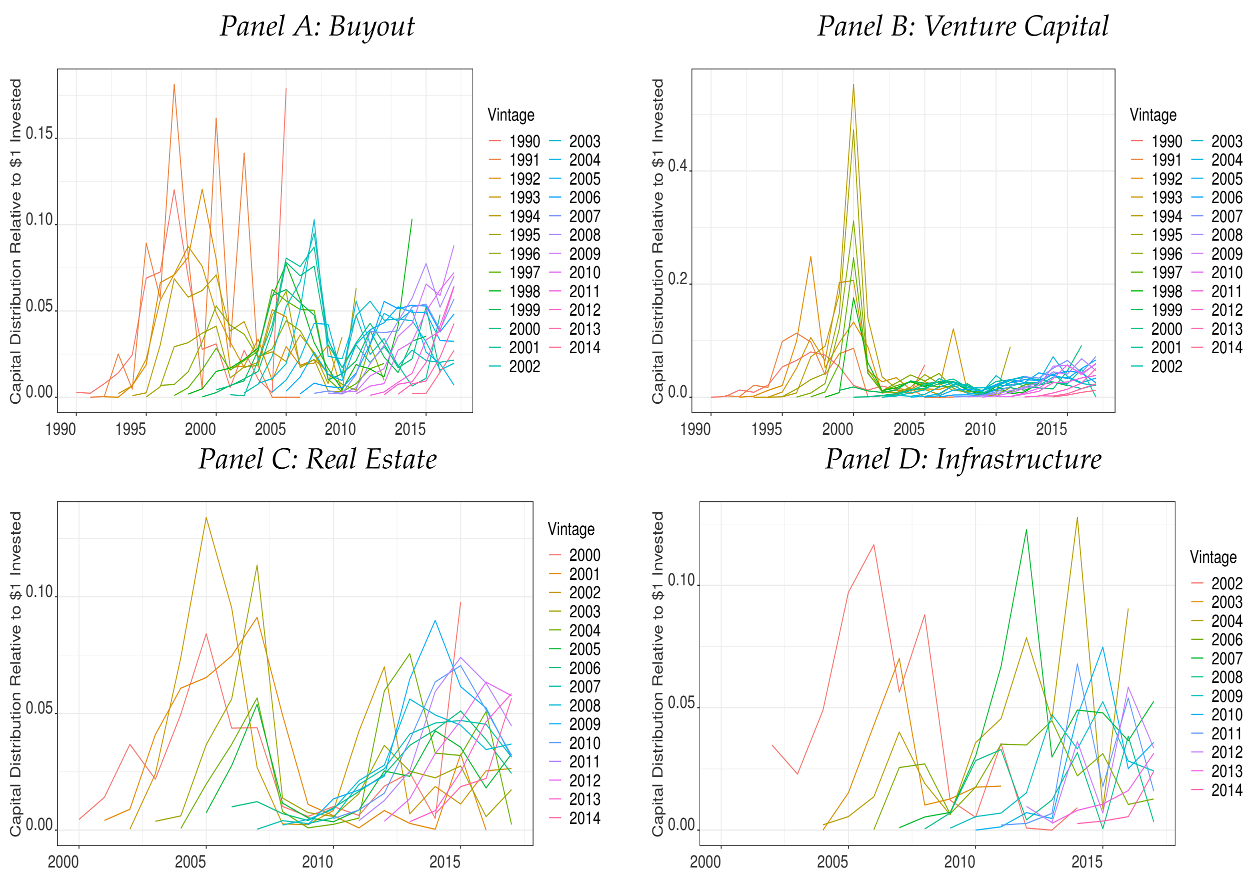
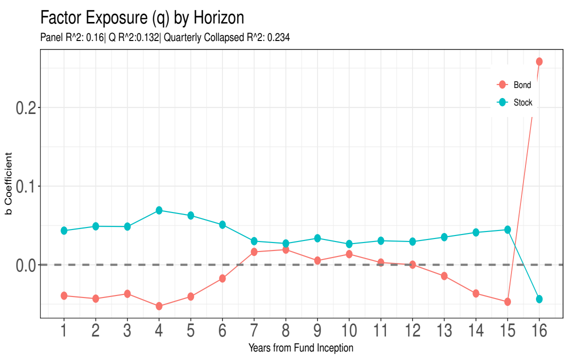
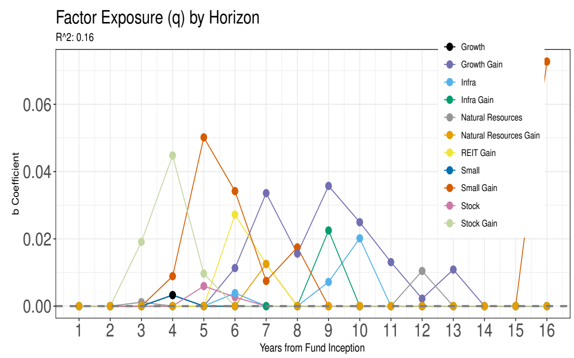


FIGURE 9: Factor Exposure over Fund Horizon

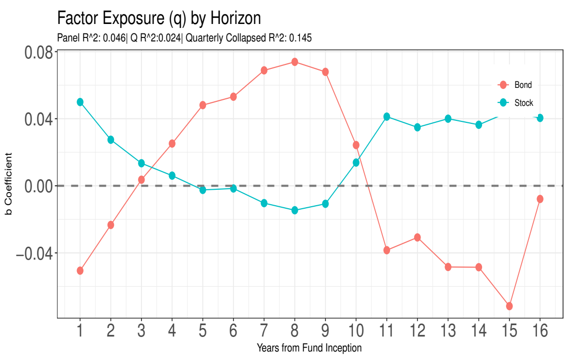
2-Factor Lasso

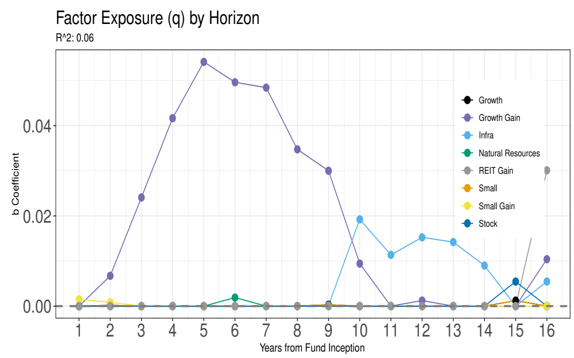
Panel A: Buyout



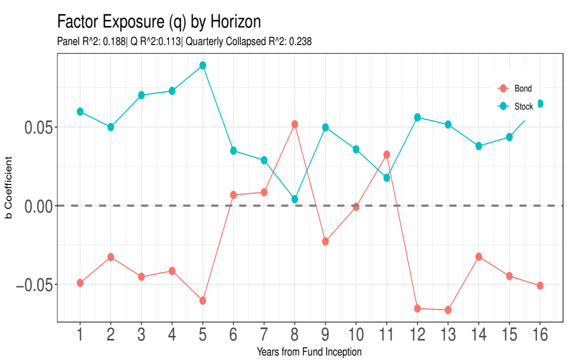


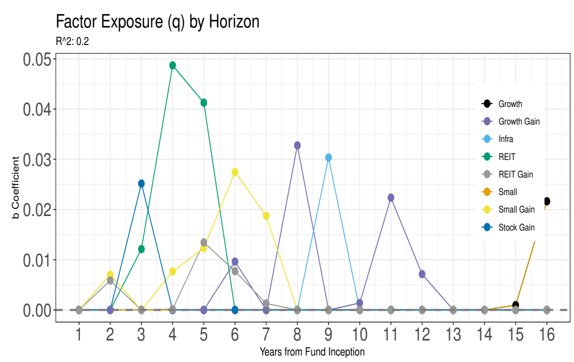
Panel B: Venture Capital



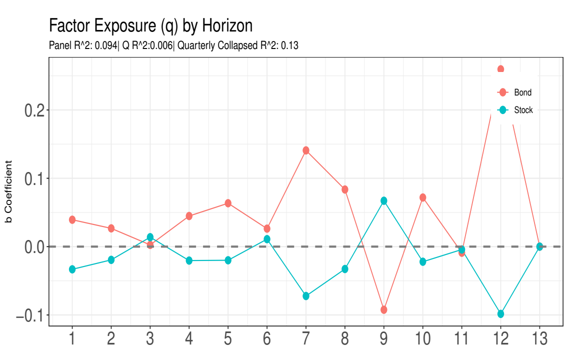


Panel C: Real Estate





Panel D: Infrastructure



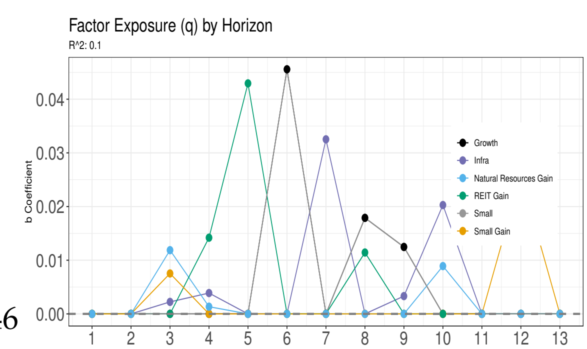
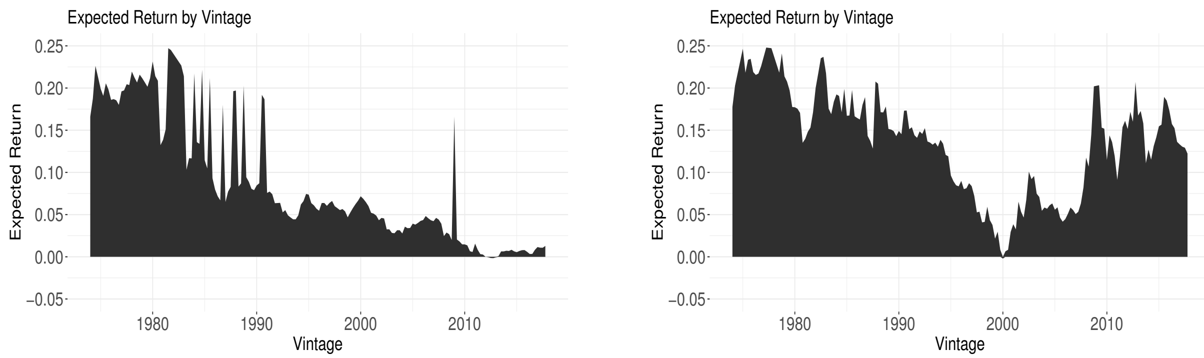


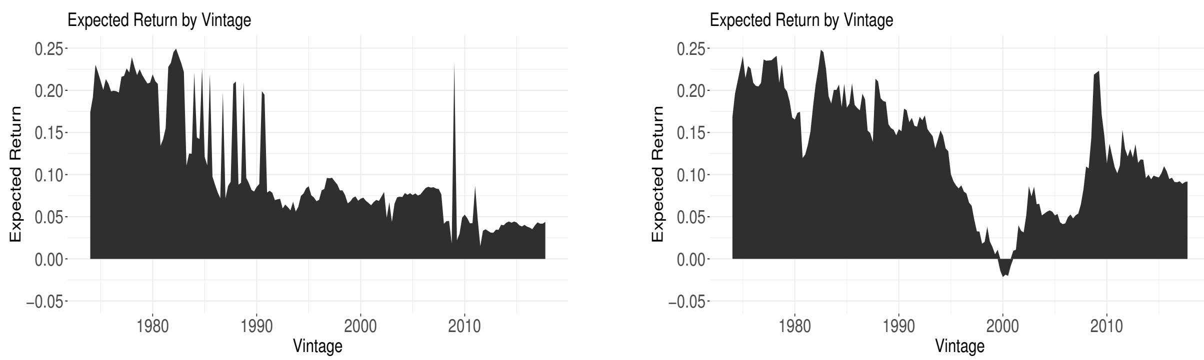
FIGURE 10: Expected Returns by Vintage

2-Factor | Lasso

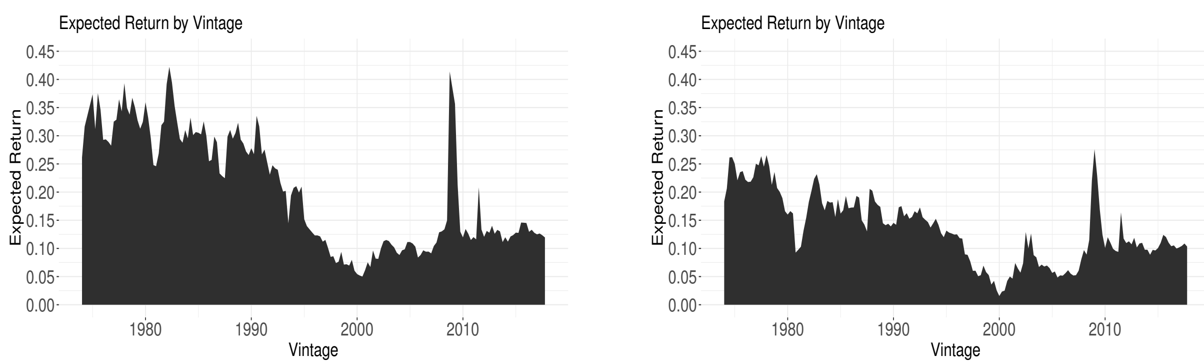
Panel A: Buyout



Panel B: Venture Capital



Panel C: Real Estate



Panel D: Infrastructure

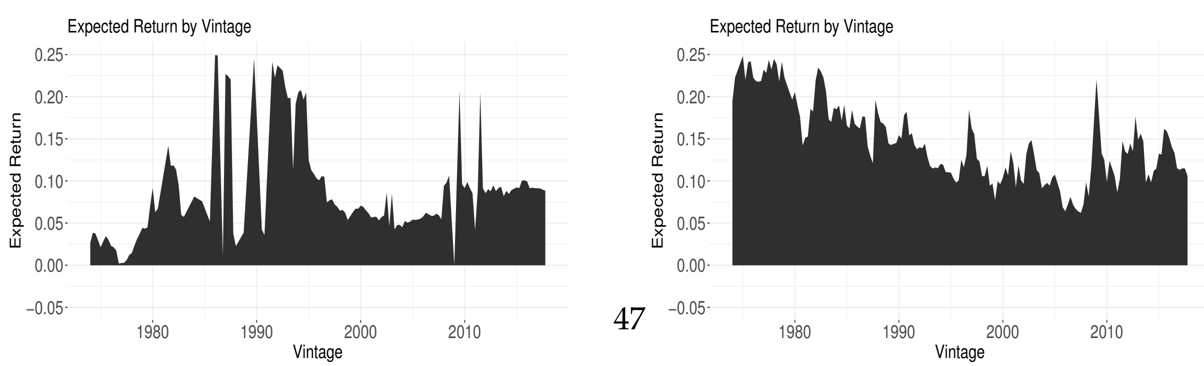
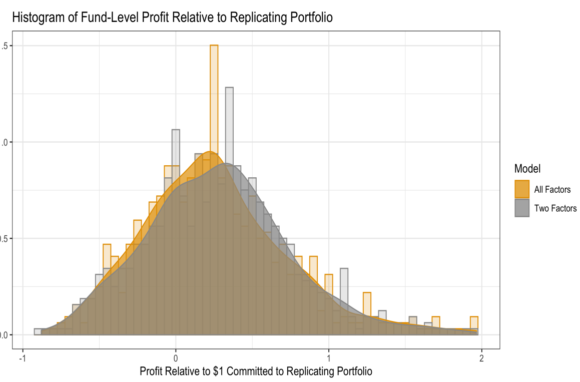


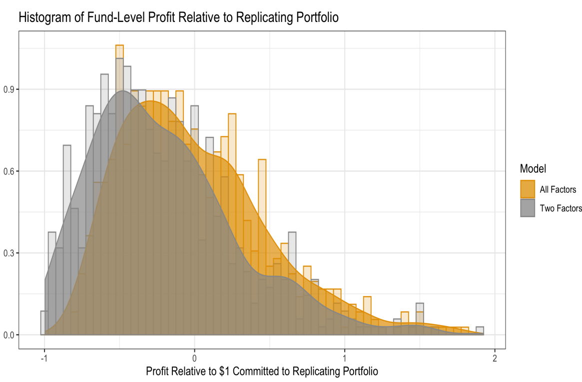
FIGURE 11: Profit Comparison

2-Factor | Lasso

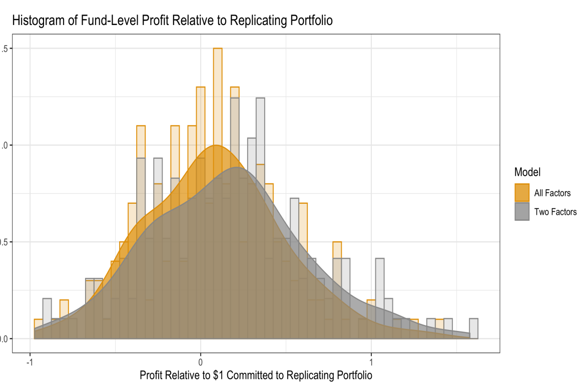
Panel A: Buyout



Panel B: Venture Capital



Panel C: Real Estate



Panel D: Infrastructure

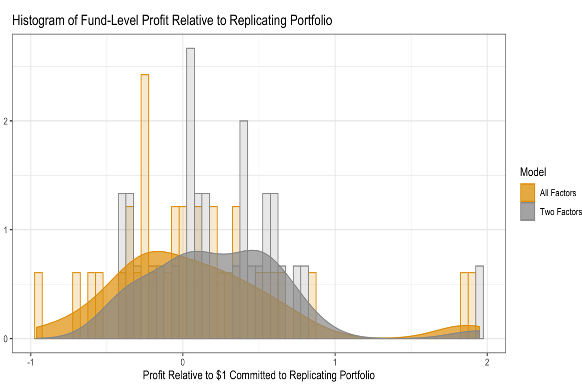
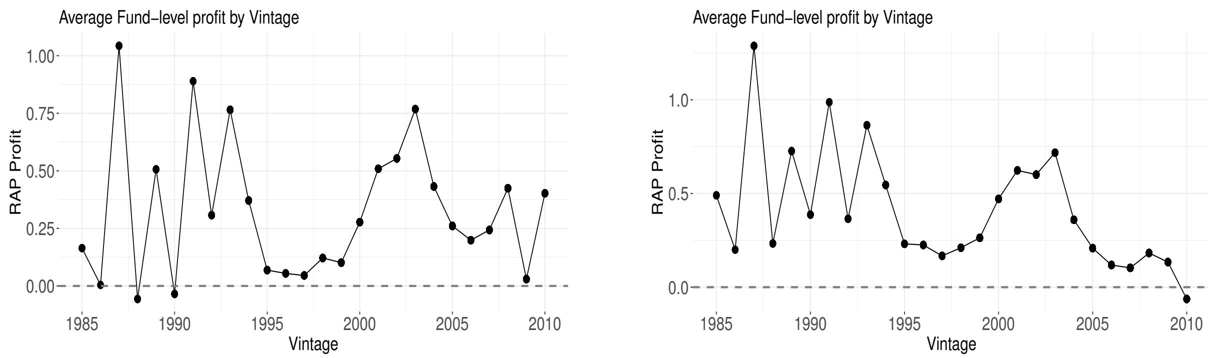


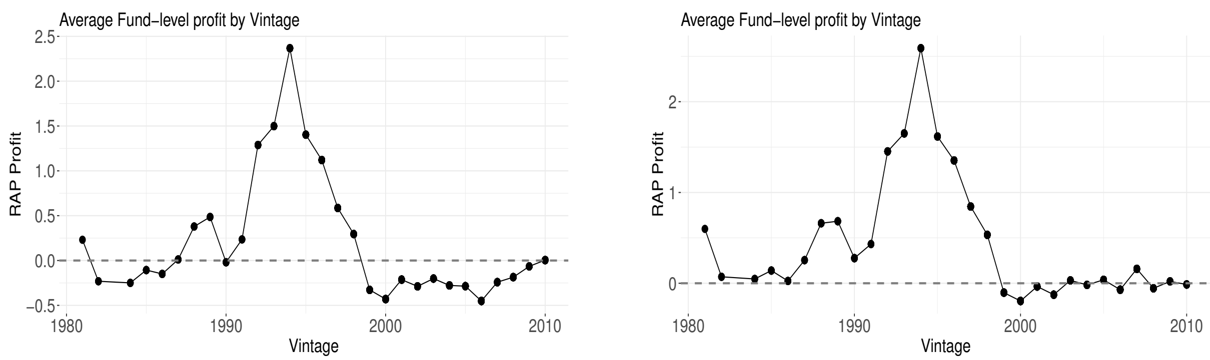
FIGURE 12: Profits Over Time

2-Factor | Lasso

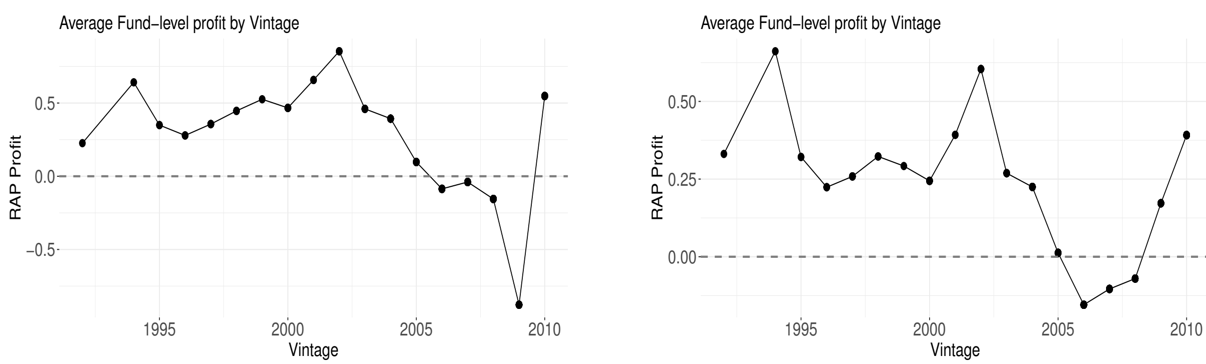
Panel A: Buyout



Panel B: Venture Capital



Panel C: Real Estate



Panel D: Infrastructure

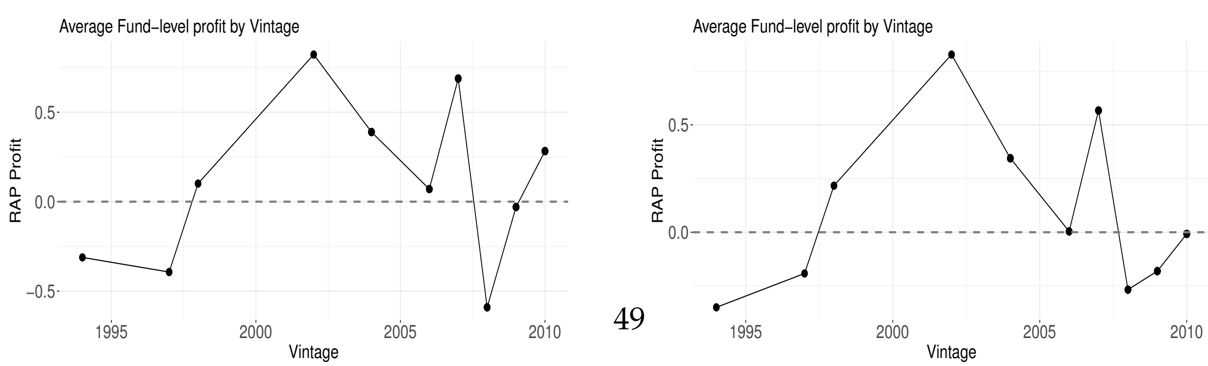
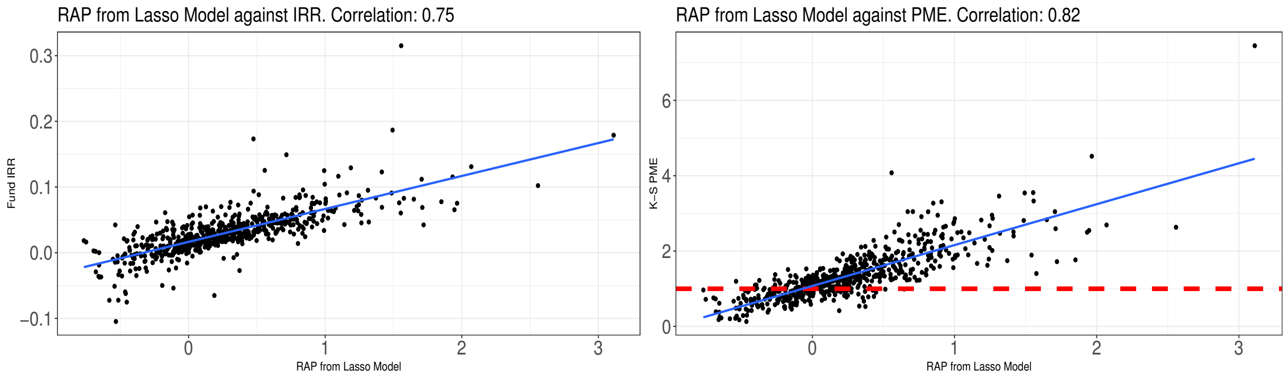
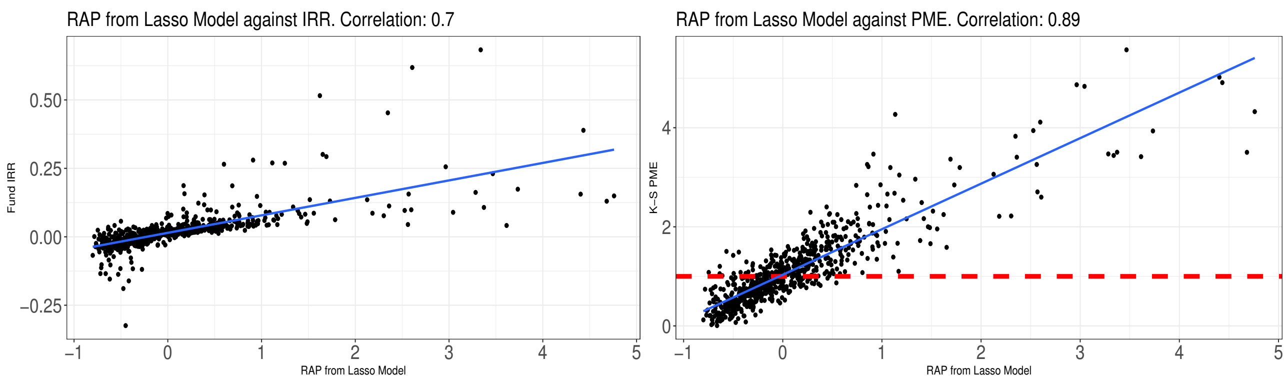


FIGURE 13: Lasso Alternative Approach Comparison

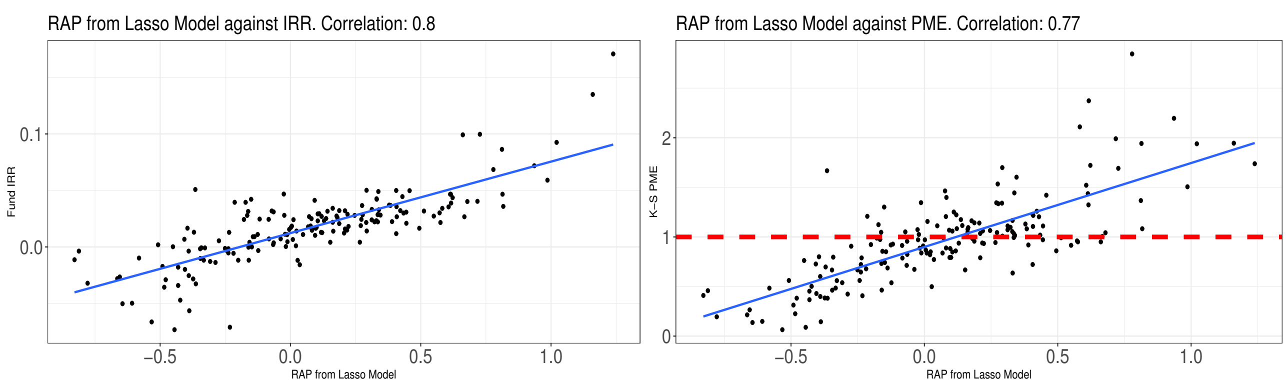
Panel A: Buyout



Panel B: Venture Capital



Panel C: Real Estate



Panel D: Infrastructure

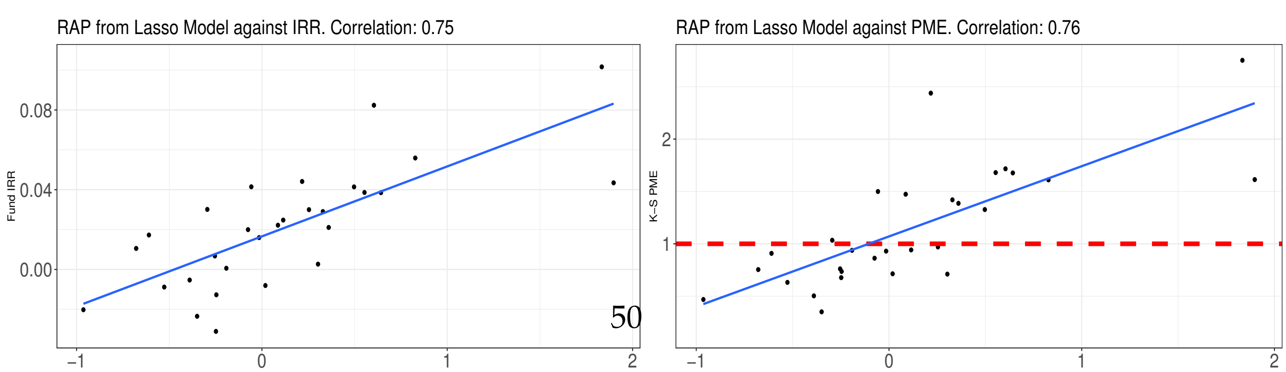


TABLE 1: Summary Statistics

Panel A: Fund Count

Vintage	Buyout	Debt Fund	Fund of Funds	Infrastructure	Real Estate	Restructuring	Venture Capital	Total	P/D Ratio
1981	0	0	0	0	0	0	1	1	1
1982	0	0	0	0	0	0	3	3	1
1983	0	0	1	0	0	0	1	2	1
1984	0	0	0	0	0	0	3	3	1
1985	4	0	0	0	0	0	6	10	1
1986	2	0	3	0	0	0	7	12	1.3
1987	5	0	0	0	0	0	5	10	1.7
1988	7	0	1	0	0	0	4	12	1
1989	3	0	0	1	0	0	5	9	1.8
1990	7	0	2	0	0	1	8	18	1.4
1991	2	0	1	0	0	2	4	9	2
1992	9	0	0	1	1	2	12	25	2
1993	9	0	2	1	0	0	11	23	2
1994	15	0	1	2	1	1	12	32	2
1995	14	0	5	1	2	0	17	39	2
1996	20	0	1	3	3	3	21	51	2.8
1997	23	0	4	2	6	2	26	63	3
1998	40	0	11	4	3	1	32	91	3
1999	31	1	9	1	2	3	47	94	3
2000	33	2	17	0	6	3	84	145	3
2001	21	0	19	1	2	5	55	103	3
2002	24	1	13	3	3	3	29	76	3
2003	18	1	12	2	7	4	20	64	3
2004	27	1	19	6	11	2	33	99	3
2005	55	2	33	5	19	6	49	169	3
2006	71	1	51	8	32	11	59	233	3.0
2007	72	1	48	13	35	13	72	254	2.9
2008	65	5	61	11	31	11	62	246	2.0
2009	27	2	30	9	12	8	26	114	1.4
2010	39	4	38	18	28	8	39	174	2.6
2011	53	2	64	22	46	12	49	248	2.5
2012	63	3	54	19	36	12	44	231	2.3
2013	63	15	66	19	59	20	53	295	2.3
2014	68	12	67	26	46	16	67	302	2.9
2015	72	16	72	21	73	18	76	348	2
2016	91	13	78	32	59	10	72	355	2
2017	47	20	39	22	61	7	60	256	2

Panel B: Fund AUM ($m)

Vintage	Buyout	Debt Fund	Fund of Funds	Infrastructure	Real Estate	Restructuring	Venture Capital	Total
1981	0	0	0	0	0	0	0	0
1982	0	0	0	0	0	0	55	55
1983	0	0	75	0	0	0	0	75
1984	0	0	0	0	0	0	189	189
1985	1,580	0	0	0	0	0	74	1,654
1986	59	0	1,510	0	0	0	293	1,862
1987	1,608	0	0	0	0	0	1,061	2,669
1988	2,789	0	0	0	0	0	463	3,252
1989	761	0	0	160	0	0	305	1,226
1990	2,457	0	1,906	0	0	153	1,125	5,641
1991	242	0	0	0	0	329	431	1,002
1992	1,150	0	0	184	0	59	1,320	2,713
1993	3,192	0	597	54	0	0	1,438	5,281
1994	6,882	0	140	1,519	488	93	1,413	10,535
1995	9,169	0	1,172	205	523	0	2,645	13,714
1996	7,435	0	242	1,114	1,851	1,600	3,820	16,062
1997	23,633	0	1,337	480	3,642	1,700	6,308	37,100
1998	38,956	0	10,879	3,933	3,461	52	8,441	65,722
1999	34,297	109	9,248	42	2,293	3,133	17,093	66,215
2000	56,299	730	13,570	0	6,757	3,320	37,802	118,478
2001	23,856	0	11,942	1,375	3,225	7,461	23,852	71,711
2002	23,705	100	10,043	1,795	5,507	1,844	7,391	50,385
2003	31,264	366	8,767	884	3,435	5,105	6,670	56,491
2004	32,855	215	5,808	5,446	6,269	2,580	8,419	61,592
2005	102,231	412	23,252	6,353	25,523	5,830	16,280	179,881
2006	215,875	778	41,728	9,726	42,442	22,928	37,285	370,762
2007	176,849	400	45,660	21,568	44,657	40,486	24,198	353,818
2008	165,982	4,697	39,493	27,228	37,789	26,158	31,839	333,186
2009	38,738	195	11,544	12,108	9,451	11,170	8,880	92,086
2010	30,612	1,684	20,662	19,891	20,125	11,855	21,457	126,286
2011	97,669	1,720	26,915	21,581	48,951	19,237	23,747	239,820
2012	94,732	733	40,384	34,467	23,417	23,943	29,566	247,242
2013	88,022	16,333	20,013	32,762	65,703	25,660	26,026	274,519
2014	126,182	5,638	37,191	50,504	32,206	21,792	35,863	309,376
2015	115,376	12,421	52,430	33,903	75,627	34,845	31,294	355,896
2016	165,779	10,721	47,705	53,388	45,416	11,368	36,433	370,810
2017	112,820	15,942	18,539	20,385	52,750	9,609	25,659	255,704

TABLE 2: Model Comparison

Model	Buyout			VC			Real Estate			Infrastructure		
	R^2	Profit	SD Profit	R^2	Profit	SD Profit	R^2	Profit	SD Profit	R^2	Profit	SD Profit
TVPI	0	0.595	(0.772)	0	0.44	(1.972)	0	0.17	(0.515)	0	0.148	(0.618)
IRR	0	12.9	(13.4)	0	8.6	(22.9)	0	8.1	(10.5)	0	9	(11.5)
PME	0	0.402	(0.674)	0	0.266	(1.624)	0	-0.01	(0.463)	0	0.164	(0.559)
2 factor	0.153	0.32	(0.487)	0.041	0.126	(1.408)	0.178	0.186	(0.416)	0.089	0.199	(0.616)
3 factor	0.155	0.385	(0.49)	0.045	0.303	(1.416)	0.192	0.095	(0.413)	0.092	0.265	(0.623)
5 factor	0.161	0.289	(0.502)	0.062	0.153	(1.428)	0.197	0.038	(0.422)	0.094	0.152	(0.623)
All Dividend Factors	0.159	0.366	(0.533)	0.053	0.107	(1.446)	0.194	-0.071	(0.477)	0.097	0.265	(0.629)
Full factors	0.16	0.266	(0.504)	0.06	0.206	(1.416)	0.2	0.081	(0.41)	0.1	0.195	(0.63)
Full factors, Quarterly	0.228	0.459	(0.413)	0.197	0.485	(1.054)	0.27	0.241	(0.364)	0.129	0.138	(0.455)

A Appendix: Asset Pricing Model

A.1 Risk-free rate

The real short yield $y_{t,1}$, or risk-free rate, satisfies $E_t[\exp\{m_{t+1}+y_{t,1}\}]=1$. Solving out this Euler equation, we get:

$$\begin{aligned} y_{t,1} &= y^{\$}_{t,1} - \mathbb{E}_t[\pi_{t+1}] - \frac{1}{2}e'_{\pi}\Sigma e_{\pi} + e'_{\pi}\Sigma^{\frac{1}{2}}\Lambda_t \\ &= y_0(1) + \left(e'_{yn} - e'_{\pi}\Psi + e'_{\pi}\Sigma^{\frac{1}{2}}\Lambda_1\right) z_t. \quad (15) \\ y_0(1) &\equiv y^{\$}_{0,1} - \pi_0 - \frac{1}{2}e'_{\pi}\Sigma e_{\pi} + e'_{\pi}\Sigma^{\frac{1}{2}}\Lambda_0. \quad (16) \end{aligned}$$

where we used the expression for the real SDF

$$\begin{aligned} m_{t+1} &= m^{\$}_{t+1} + \pi_{t+1} \\ &= -y^{\$}_{t,1} - \frac{1}{2}\Lambda'_t\Lambda_t - \Lambda'_t\varepsilon_{t+1} + \pi_0 + e'_{\pi}\Psi z_t + e'_{\pi}\Sigma^{\frac{1}{2}}\varepsilon_{t+1} \\ &= -y_{t,1} - \frac{1}{2}e'_{\pi}\Sigma e_{\pi} + e'_{\pi}\Sigma^{\frac{1}{2}}\Lambda_t - \frac{1}{2}\Lambda'_t\Lambda_t - \left(\Lambda'_t - e'_{\pi}\Sigma^{\frac{1}{2}}\right)\varepsilon_{t+1} \end{aligned}$$

The real short yield is the nominal short yield minus expected inflation minus a Jensen adjustment minus the inflation risk premium.

A.2 Nominal and real term structure

Proposition 1. Nominal bond yields are affine in the state vector:

$$y^{\$}_t(\tau) = -\frac{A^{\$}_{\tau}}{\tau} - \frac{B^{\$\prime}_{\tau}}{\tau} z_t,$$

where the coefficients $A^{\$}_{\tau}$ and $B^{\$}_{\tau}$ satisfy the following recursions:

$$\begin{aligned} A^{\$}_{\tau+1} &= -y^{\$}_{0,1} + A^{\$}_{\tau} + \frac{1}{2}\left(B^{\$}_{\tau}\right)'\Sigma\left(B^{\$}_{\tau}\right) - \left(B^{\$}_{\tau}\right)'\Sigma^{\frac{1}{2}}\Lambda_0, \quad (17) \\ \left(B^{\$}_{\tau+1}\right)' &= \left(B^{\$}_{\tau}\right)'\Psi - e'_{yn} - \left(B^{\$}_{\tau}\right)'\Sigma^{\frac{1}{2}}\Lambda_1, \quad (18) \end{aligned}$$

initialized at $A^{\$}_0 = 0$ and $B^{\$}_0 = 0$.

Proof. We conjecture that the $t+1$-price of a τ-period bond is exponentially affine in the state:

$$\log(P^{\$}_{t+1,\tau}) = A^{\$}_{\tau} + \left(B^{\$}_{\tau}\right)' z_{t+1}$$

and solve for the coefficients $A^{\$}_{\tau+1}$ and $B^{\$}_{\tau+1}$ in the process of verifying this conjecture using the Euler equation:

$$\begin{aligned}
P^{\$}_{t,\tau+1} &= \mathbb{E}_t[\exp\{m^{\$}_{t+1} + \log\left(P^{\$}_{t+1,\tau}\right)\}] \\
&= \mathbb{E}_t[\exp\{-y^{\$}_{t,1} - \frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\varepsilon_{t+1} + A^{\$}_{\tau} + \left(B^{\$}_{\tau}\right)' z_{t+1}\}] \\
&= \exp\{-y^{\$}_{0,1} - e_{yn}' z_t - \frac{1}{2}\Lambda_t'\Lambda_t + A^{\$}_{\tau} + \left(B^{\$}_{\tau}\right)' \Psi z_t\} \times \\
&\quad \mathbb{E}_t\left[\exp\{-\Lambda_t'\varepsilon_{t+1} + \left(B^{\$}_{\tau}\right)' \Sigma^{\frac{1}{2}}\varepsilon_{t+1}\}\right].
\end{aligned}$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for Λ_t to get:

$$P^{\$}_{t,\tau+1} = \exp\left\{-y^{\$}_{0,1} - e_{yn}' z_t + A^{\$}_{\tau} + \left(B^{\$}_{\tau}\right)' \Psi z_t + \frac{1}{2}\left(B^{\$}_{\tau}\right)' \Sigma \left(B^{\$}_{\tau}\right) - \left(B^{\$}_{\tau}\right)' \Sigma^{\frac{1}{2}}\left(\Lambda_0 + \Lambda_1 z_t\right)\right\}.$$

Taking logs and collecting terms, we obtain a linear equation for $\log(p_t(\tau+1))$:

$$\log\left(P^{\$}_{t,\tau+1}\right) = A^{\$}_{\tau+1} + \left(B^{\$}_{\tau+1}\right)' z_t,$$

where $A^{\$}_{\tau+1}$ satisfies (17) and $B^{\$}_{\tau+1}$ satisfies (18). The relationship between log bond prices and bond yields is given by $-\log\left(P^{\$}_{t,\tau}\right)/\tau = y^{\$}_{t,\tau}$. □

Define the one-period return on a nominal zero-coupon bond as:

$$r^{b,\$}_{t+1,\tau} = \log\left(P^{\$}_{t+1,\tau}\right) - \log\left(P^{\$}_{t,\tau+1}\right)$$

The nominal bond risk premium on a bond of maturity τ is the expected excess return corrected for a Jensen term, and equals negative the conditional covariance between that bond return and the nominal SDF:

$$\begin{aligned}
\mathbb{E}_t\left[r^{b,\$}_{t+1,\tau}\right] - y^{\$}_{t,1} + \frac{1}{2}\mathbb{V}_t\left[r^{b,\$}_{t+1,\tau}\right] &= -\mathbb{C}_t\left[m^{\$}_{t+1}, r^{b,\$}_{t+1,\tau}\right] \\
&= \left(B^{\$}_{\tau}\right)' \Sigma^{\frac{1}{2}}\Lambda_t
\end{aligned}$$

Real bond yields, $y_{t,\tau}$, denoted without the \$ superscript, are affine as well with coefficients that follow similar recursions:

$$A_{\tau+1} = -y_{0,1} + A_{\tau} + \frac{1}{2}B_{\tau}'\Sigma B_{\tau} - B_{\tau}'\Sigma^{\frac{1}{2}}\left(\Lambda_0 - \Sigma^{\frac{1}{2}\prime} e_{\pi}\right), \quad (19)$$

$$B_{\tau+1}' = -e_{yn}' + \left(e_{\pi} + B_{\tau}\right)'\left(\Psi - \Sigma^{\frac{1}{2}}\Lambda_1\right). \quad (20)$$

For $\tau = 1$, we recover the expression for the risk-free rate in (15)-(16).

A.3 Stock Market

We define the real return on equity as $R^m_{t+1} = \frac{P^m_{t+1}+D^m_{t+1}}{P^m_t}$, where P^m_t is the end-of-period price on the equity market. A log-linearization delivers:

$$r^m_{t+1} = \kappa^m_0 + \Delta d^m_{t+1} + \kappa^m_1 pd^m_{t+1} - pd^m_t. \tag{21}$$

The unconditional mean log real stock return is $r^m_0 = \mathbb{E}[r^m_t]$, the unconditional mean dividend growth rate is $\mu^m = \mathbb{E}[\Delta d^m_{t+1}]$, and $\overline{pd^m} = \mathbb{E}[pd^m_t]$ is the unconditional average log price-dividend ratio on equity. The linearization constants κ^m_0 and κ^m_1 are defined as:

$$\kappa^m_1 = \frac{e^{\overline{pd^m}}}{e^{\overline{pd^m}}+1} < 1 \text{ and } \kappa^m_0 = \log\left(e^{\overline{pd^m}}+1\right) - \frac{e^{\overline{pd^m}}}{e^{\overline{pd^m}}+1}\overline{pd^m}. \tag{22}$$

Our state vector z contains the (demeaned) log real dividend growth rate on the stock market, $\Delta d^m_{t+1} - \mu^m$, and the (demeaned) log price-dividend ratio $pd^m - \overline{pd^m}$.

$$\begin{aligned} pd^m_t(\tau) &= \overline{pd^m} + e'_{pd} z_t, \\ \Delta d^m_t &= \mu^m + e'_{divm} z_t, \end{aligned}$$

where e'_{pd} (e_{divm}) is a selector vector that has a one in the fifth (sixth) entry, since the log pd ratio (log dividend growth rate) is the fifth (sixth) element of the VAR.

We *define* the log return on the stock market so that the return equation holds exactly, given the time series for $\{\Delta d^m_t, pd^m_t\}$. Rewriting (21):

$$\begin{aligned} r^m_{t+1} - r^m_0 &= \left[(e_{divm} + \kappa^m_1 e_{pd})'\Psi - e'_{pd}\right] z_t + \left(e_{divm} + \kappa^m_1 e_{pd}\right)' \Sigma^{\frac{1}{2}} \varepsilon_{t+1}. \\ r^m_0 &= \mu^m + \kappa^m_0 - \overline{pd^m}(1-\kappa^m_1). \end{aligned}$$

The equity risk premium is the expected excess return on the stock market corrected for a Jensen term. By the Euler equation, it equals minus the conditional covariance between the log SDF and the log return:

$$\begin{aligned} 1 &= \mathbb{E}_t\left[M_{t+1}\frac{P^m_{t+1}+D^m_{t+1}}{P^m_t}\right] = E_t\left[\exp\{m^\$_{t+1} + \pi_{t+1} + r^m_{t+1}\}\right] \\ &= \mathbb{E}_t\left[\exp\left\{-y^\$_{t,1} - \frac{1}{2}\Lambda'_t\Lambda_t - \Lambda'_t\varepsilon_{t+1} + \pi_0 + e'_\pi z_{t+1} + r^m_0 + (e_{divm} + \kappa^m_1 e_{pd})' z_{t+1} - e'_{pd} z_t\right\}\right] \\ &= \exp\left\{-y^\$_0(1) - \frac{1}{2}\Lambda'_t\Lambda_t + \pi_0 + r^m_0 + \left[(e_{divm} + \kappa^m_1 e_{pd} + e_\pi)'\Psi - e'_{pd} - e'_{yn}\right] z_t\right\} \\ &\quad \times \mathbb{E}_t\left[\exp\{-\Lambda'_t\varepsilon_{t+1} + \left(e_{divm} + \kappa^m_1 e_{pd} + e_\pi\right)' \Sigma^{\frac{1}{2}}\varepsilon_{t+1}\right] \\ &= \exp\left\{r^m_0 + \pi_0 - y^\$_0(1) + \left[(e_{divm} + \kappa^m_1 e_{pd} + e_\pi)'\Psi - e'_{pd} - e'_{yn}\right] z_t\right\} \\ &\quad \times \exp\left\{\frac{1}{2}\left(e_{divm} + \kappa^m_1 e_{pd} + e_\pi\right)' \Sigma \left(e_{divm} + \kappa^m_1 e_{pd} + e_\pi\right) - \left(e_{divm} + \kappa^m_1 e_{pd} + e_\pi\right)' \Sigma^{\frac{1}{2}}\Lambda_t\right\} \end{aligned}$$

Taking logs on both sides delivers:

$$
\begin{aligned}
r_0^m + \pi_0 - y_0^{\$}(1) + \left[(e_{divm} + \kappa_1^m e_{pd} + e_\pi)'\Psi - e_{pd}' - e_{yn}' \right] z_t & \\
+\frac{1}{2}\left(e_{divm} + \kappa_1^m e_{pd} + e_\pi\right)' \Sigma \left(e_{divm} + \kappa_1^m e_{pd} + e_\pi\right) &= \left(e_{divm} + \kappa_1^m e_{pd} + e_\pi\right)' \Sigma^{\frac{1}{2}} \Lambda_t \\
\mathbb{E}_t \left[r_{t+1}^{m,\$} \right] - y_{t,1}^{\$} + \frac{1}{2}\mathbb{V}_t \left[r_{t+1}^{m,\$} \right] &= -\mathbb{C}_t \left[m_{t+1}^{\$}, r_{t+1}^{m,\$} \right]
\end{aligned} \tag{23}
$$

The left-hand side is the expected excess return on the stock market, corrected for a Jensen term, while the right-hand side is the negative of the conditional covariance between the (nominal) log stock return and the nominal log SDF. We refer to the left-hand side as the equity risk premium in the data, since it is implied directly by the VAR. We refer to the right-hand side as the equity risk premium in the model, since it requires knowledge of the market prices of risk.

Note that we can obtain the same expression using the log real SDF and log real stock return:

$$
\begin{aligned}
\mathbb{E}_t \left[r_{t+1}^m \right] - y_{t,1} + \frac{1}{2}\mathbb{V}_t \left[r_{t+1}^m \right] &= -\mathbb{C}_t \left[m_{t+1}, r_{t+1}^m \right] \\
r_0^m - y_0(1) &+ \left[(e_{divm} + \kappa_1^m e_{pd} + e_\pi)'\Psi - e_{pd}' - e_{yn}' - e_\pi' \Sigma^{1/2}\Lambda_1 \right] z_t \\
+\frac{1}{2}(e_{divm} + \kappa_1^m e_{pd})'\Sigma(e_{divm} + \kappa_1^m e_{pd}) &= \left(e_{divm} + \kappa_1^m e_{pd}\right)' \Sigma^{1/2}(\Lambda_t - \left(\Sigma^{1/2}\right)' e_\pi)
\end{aligned}
$$

We combine the terms in Λ_0 and Λ_1 on the right-hand side and plug in for $y_0(1)$ from (16) to get:

$$
\begin{aligned}
r_0^m + \pi_0 - y_{0,1}^{\$} + \frac{1}{2} e_\pi' \Sigma e_\pi \qquad & +\frac{1}{2}(e_{divm} + \kappa_1^m e_{pd})'\Sigma(e_{divm} + \kappa_1^m e_{pd}) + e_\pi' \Sigma \left(e_{divm} + \kappa_1^m e_{pd}\right) \\
+ \left[(e_{divm} + \kappa_1^m e_{pd} + e_\pi)'\Psi - e_{pd}' - e_{yn}' \right] z_t &= \left(e_{divm} + \kappa_1^m e_{pd}\right)' \Sigma^{1/2}\Lambda_t + e_\pi' \Sigma^{\frac{1}{2}} \Lambda_0 + e_\pi' \Sigma^{1/2}\Lambda_1 z_t
\end{aligned}
$$

This recovers equation (23).

A.4 Dividend Strips

A.4.1 Affine Structure for Price-Dividend Ratio on Equity Strip

Proposition 2. Log price-dividend ratios on dividend strips are affine in the state vector:

$$
p_{t,\tau}^d = \log\left(P_{t,\tau}^d\right) = A_\tau^m + B_\tau^{m\prime} z_t,
$$

where the coefficients A_τ^m and B_τ^m follow recursions:

$$
\begin{aligned}
A_{\tau+1}^m &= A_\tau^m + \mu_m - y_0(1) + \frac{1}{2}\left(e_{divm} + B_\tau^m\right)' \Sigma \left(e_{divm} + B_\tau^m\right) \\
&\quad - \left(e_{divm} + B_\tau^m\right)' \Sigma^{\frac{1}{2}} \left(\Lambda_0 - \Sigma^{\frac{1}{2}\prime} e_\pi\right), \qquad (24) \\
B_{\tau+1}^{m\prime} &= \left(e_{divm} + e_\pi + B_\tau^m\right)' \Psi - e_{yn}' - \left(e_{divm} + e_\pi + B_\tau^m\right)' \Sigma^{\frac{1}{2}} \Lambda_1, \qquad (25)
\end{aligned}
$$

initialized at $A_0^m = 0$ and $B_0^m = 0$.

Proof. We conjecture the affine structure and solve for the coefficients $A^m_{\tau+1}$ and $B^m_{\tau+1}$ in the process of verifying this conjecture using the Euler equation:

$$\begin{aligned} P^d_{t,\tau+1} &= \mathbb{E}_t\left[M_{t+1}P^d_{t+1,\tau}\frac{D^m_{t+1}}{D^m_t}\right] = E_t\left[\exp\{m^\$_{t+1}+\pi_{t+1}+\Delta d^m_{t+1}+p^d_{t+1}(\tau)\}\right] \\ &= \mathbb{E}_t\left[\exp\{-y^\$_{t,1}-\frac{1}{2}\Lambda_t'\Lambda_t-\Lambda_t'\varepsilon_{t+1}+\pi_0+e_\pi'z_{t+1}+\mu^m+e_{divm}'z_{t+1}+A^m_\tau+B^{m\prime}_\tau z_{t+1}\}\right] \\ &= \exp\{-y^\$_0(1)-e_{yn}'z_t-\frac{1}{2}\Lambda_t'\Lambda_t+\pi_0+e_\pi'\Psi z_t+\mu_m+e_{divm}'\Psi z_t+A^m_\tau+B^{m\prime}_\tau\Psi z_t\} \\ &\quad\times\mathbb{E}_t\left[\exp\{-\Lambda_t'\varepsilon_{t+1}+(e_{divm}+e_\pi+B^m_\tau)'\,\Sigma^{\frac{1}{2}}\varepsilon_{t+1}\right]. \end{aligned}$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for Λ_t to get:

$$\begin{aligned} P^d_{t,\tau+1} &= \exp\{-y^\$_0(1)+\pi_0+\mu_m+A^m_\tau+\left[(e_{divm}+e_\pi+B^m_\tau)'\,\Psi-e_{yn}'\right]z_t \\ &\quad+\frac{1}{2}\left(e_{divm}+e_\pi+B^m_\tau\right)'\Sigma\left(e_{divm}+e_\pi+B^m_\tau\right) \\ &\quad-\left(e_{divm}+e_\pi+B^m_\tau\right)'\Sigma^{\frac{1}{2}}\left(\Lambda_0+\Lambda_1 z_t\right)\} \end{aligned}$$

Taking logs and collecting terms, we obtain a log-linear expression for $p^d_t(\tau+1)$:

$$p^d_{t,\tau+1} = A^m_{\tau+1}+B^{m\prime}_{\tau+1}z_t,$$

where:

$$\begin{aligned} A^m_{\tau+1} &= A^m_\tau+\mu_m-y^\$_0(1)+\pi_0+\frac{1}{2}\left(e_{divm}+e_\pi+B^m_\tau\right)'\Sigma\left(e_{divm}+e_\pi+B^m_\tau\right) \\ &\quad-\left(e_{divm}+e_\pi+B^m_\tau\right)'\Sigma^{\frac{1}{2}}\Lambda_0, \\ B^{m\prime}_{\tau+1} &= \left(e_{divm}+e_\pi+B^m_\tau\right)'\Psi-e_{yn}'-\left(e_{divm}+e_\pi+B^m_\tau\right)'\Sigma^{\frac{1}{2}}\Lambda_1. \end{aligned}$$

We recover the recursions in (24) and (25) after using equation (16). □

Like we did for the stock market as a whole, we define the strip risk premium as:

$$\begin{aligned} \mathbb{E}_t\left[r^{d,\$}_{t+1,\tau}\right]-y^\$_{t,1}+\frac{1}{2}\mathbb{V}_t\left[r^{d,\$}_{t+1,\tau}\right] &= -\mathbb{C}_t\left[m^\$_{t+1},r^{d,\$}_{t+1,\tau}\right] \\ &= \left(e_{divm}+e_\pi+B^m_\tau\right)'\Sigma^{\frac{1}{2}}\Lambda_t \end{aligned}$$

The risky strips for REITs, infrastructure, small stocks, growth stocks, and natural resources are defined analogously.

A.4.2 Expected Holding Period Return over k-horizons on Dividend Strips

The expected return on a dividend strip that pays the realized nominal dividend k quarters hence and that is held to maturity is:

$$\begin{aligned}
\mathbb{E}_t[R_{t\to t+k}] &= \frac{\mathbb{E}_t\left[\frac{D^\$_{t+k}}{D^\$_t}\right]}{P^d_{t,k}} - 1 \\
&= \exp\left(-A^m_k - B^{m\prime}_k z_t + \mathbb{E}_t\left[\sum_{s=1}^{k}\Delta d_{t+s} + \pi_{t+s}\right] + \frac{1}{2}\mathbb{V}_t\left[\sum_{s=1}^{k}\Delta d_{t+s} + \pi_{t+s}\right]\right) - 1 \qquad (26)\\
&= \exp\left(-A^m_k - B^{m\prime}_k z_t + k(\mu_m + \pi_0) + (e_{divm} + e_\pi)'\left[\sum_{s=1}^{k}\Psi^s\right] z_t + \frac{k}{2}(e_{divm} + e_\pi)'\Sigma(e_{divm} + e_\pi)\right) - 1
\end{aligned}$$

These are the building blocks for computing the expected return on a PE investment.

A.4.3 Dividend Futures: Price and Return

The price of a dividend futures contract which delivers one quarter worth of nominal dividends at quarter $t+\tau$, divided by the current dividend, is equal to:

$$\frac{F^d_{t,\tau}}{D^\$_t} = P^d_{t,\tau}\exp\left(\tau y^\$_{t,\tau}\right),$$

where $P^d_{t,\tau}$ is the spot price-dividend ratio. Using the affine expressions for the strip price-dividend ratio and the nominal bond price, it can be written as:

$$\frac{F^d_{t,\tau}}{D^\$_t} = \exp\left(A^m_\tau - A^\$_\tau + (B^m_\tau - B^\$_\tau)' z_t\right),$$

The one-period holding period return on the dividend future of maturity τ is:

$$R^{fut,d}_{t+1,\tau} = \frac{F^d_{t+1,\tau-1}}{F^d_{t,\tau}} - 1 = \frac{F^d_{t+1,\tau-1}/D^\$_{t+1}}{F^d_{t,\tau}/D^\$_t}\frac{D^\$_{t+1}}{D^\$_t} - 1$$

It can be written as:

$$\begin{aligned}
\log\left(1 + R^{fut,d}_{t+1,\tau}\right) &= A^m_{\tau-1} - A^\$_{\tau-1} - A^m_\tau + A^\$_\tau + \mu_m + \pi_0 \\
&\quad +(B^m_{\tau-1} - B^\$_{\tau-1} + e_{divm} + e_\pi)' z_{t+1} - (B^m_\tau - B^\$_\tau)' z_t
\end{aligned}$$

The expected log return, which is already a risk premium on account of the fact that the dividend future already takes out the return on an equal-maturity nominal Treasury bond, equals:

$$\begin{aligned}
\mathbb{E}_t\left[\log\left(1 + R^{fut,d}_{t+1,\tau}\right)\right] &= A^m_{\tau-1} - A^\$_{\tau-1} - A^m_\tau + A^\$_\tau + \mu_m + \pi_0 \\
&\quad + \left[(B^m_{\tau-1} - B^\$_{\tau-1} + e_{divm} + e_\pi)'\Psi - (B^m_\tau - B^\$_\tau)'\right] z_t
\end{aligned}$$

Given that the state variable z_t is mean-zero, the first row denotes the unconditional dividend futures risk premium.

A.5 Capital Gain Strips

A capital gain strip is a strip that pays the realized ex-dividend stock price P^m_{t+k} at time $t+k$. For convenience, we scale this payout by the current stock price P^m_t. In other words, the claim pays off the realized cumulative capital gain between periods t and $t+k$, $\frac{P^m_{t+k}}{P^m_t}$.

By value additivity of the dividend strips, the time-t price of this claim is today's stock price minus the prices of the dividend strips of horizons $1,\cdots,k$:

$$\frac{P^m_t - \left(P^d_{t,1} + \cdots + P^d_{t,k}\right) D^m_t}{P^m_t} = 1 - \frac{P^d_{t,1} + \cdots + P_{t,k}}{P^m_t / D^m_t} = 1 - \frac{\sum_{\tau=1}^k \exp\{A^m_\tau + (B^m_\tau)' z_t\}}{\exp\{\overline{pd^m} + e'_{pdm} z_t\}}$$

The expected return on the capital gains strip is given by

$$\begin{aligned}
\frac{\mathbb{E}_t\left[P^m_{t+k}\right]}{P^m_t - \left(P^d_{t,1} + \cdots + P^d_{t,k}\right) D^m_t} &= \frac{\mathbb{E}_t\left[\frac{P^m_{t+k}}{P^m_t}\right]}{1 - \frac{\sum_{\tau=1}^k \exp\{A^m_\tau + (B^m_\tau)' z_t\}}{\exp\{\overline{pd^m} + e'_{pdm} z_t\}}} = \frac{\mathbb{E}_t\left[\frac{P^m_{t+k}/D^m_{t+k}}{P^m_t/D^m_t}\frac{D^m_{t+k}}{D^m_t}\right]}{1 - \frac{\sum_{\tau=1}^k \exp\{A^m_\tau + (B^m_\tau)' z_t\}}{\exp\{\overline{pd^m} + e'_{pdm} z_t\}}} \\
&= \frac{\mathbb{E}_t\left[\exp\{e'_{pdm}(z_{t+k} - z_t) + \Delta d^\$_{t+k} + \cdots + \Delta d^\$_{t+1}\}\right]}{1 - \frac{\sum_{\tau=1}^k \exp\{A^m_\tau + (B^m_\tau)' z_t\}}{\exp\{\overline{pd^m} + e'_{pdm} z_t\}}} \\
&= \frac{\mathbb{E}_t\left[\exp\{e'_{pdm}(z_{t+k} - z_t) + k(\mu_m + \pi_0) + \sum_{\tau=1}^k (e_{divm} + e_\pi)' z_{t+\tau}\}\right]}{1 - \frac{\sum_{\tau=1}^k \exp\{A^m_\tau + (B^m_\tau)' z_t\}}{\exp\{\overline{pd^m} + e'_{pdm} z_t\}}} \\
&= \frac{\exp\left\{k(\mu_m + \pi_0) + \left[e'_{pdm}(\Psi^k - I) + (e_{divm} + e_\pi)' \sum_{\tau=1}^k (\Psi)^\tau\right] z_t + \frac{1}{2}\mathcal{V}\right\}}{1 - \frac{\sum_{\tau=1}^k \exp\{A^m_\tau + (B^m_\tau)' z_t\}}{\exp\{\overline{pd^m} + e'_{pdm} z_t\}}}
\end{aligned}$$

where

$$\mathcal{V} = e'_{pdm}\left[\sum_{\tau=1}^k \Psi^{\tau-1}\Sigma(\Psi^{\tau-1})'\right] e_{pdm} + (e_{divm} + e_\pi)' \left[\sum_{\tau=1}^k \sum_{n=1}^\tau \Psi^{n-1}\Sigma\Psi^{n-1}\right](e_{divm} + e_\pi)$$

A.6 Covariance PE with stock and bond returns

The covariance of PE fund returns with stock returns and with bond returns is given by the covariance of the PE fund's replication portfolio return with stock and bond returns. The return on the replicating portfolio is the weighted average of the return on the strips that make up the portfolio. The weights are described in equation (3), where $\boldsymbol{w}^i$ is a $1 \times HK$ vector with generic element $w^i_{t,h,k} = q^i_{t,h}(k) P_{t,h}(k)$. The weights sum to one. We focus on the one-period, conditional stock beta $\beta^i_{t,m}$ and bond beta $\beta^i_{t,b}$ of PE fund

i. For the bond beta, we focus on the beta with the nominal five-year bond ($\tau = 20$).

$$\begin{aligned}
\beta^i_{t,m} &= \frac{\mathbb{C}_t(r^i_{t+1}, r^m_{t+1})}{\mathbb{V}_t(r^m_{t+1})} = \frac{\mathbb{C}_t(\sum_{h,k}^{HK} w^i_{t,h,k} r_{t+1,h,k}, r^m_{t+1})}{\mathbb{V}_t(r^m_{t+1})} = \frac{\sum_{h,k}^{HK} w^i_{t,h,k} \mathbb{C}_t(r_{t+1,h}(k), r^m_{t+1})}{\mathbb{V}_t(r^m_{t+1})} \\
&= \frac{\sum_{h,k}^{HK} w^i_{t,h,k} \left(e_{strip,k,h}\right)' \Sigma \left(e_{divm} + e_\pi + \kappa^m_1 e_{pd}\right)}{\left(e_{divm} + e_\pi + \kappa^m_1 e_{pd}\right)' \Sigma \left(e_{divm} + e_\pi + \kappa^m_1 e_{pd}\right)} \\
\beta^i_{t,b} &= \frac{\sum_{h,k}^{HK} w^i_{t,h}(k) \left(e_{strip,k,h}\right)' \Sigma \left(B^\$_{20}\right)}{\left(B^\$_{20}\right)' \Sigma \left(B^\$_{20}\right)}
\end{aligned}$$

For example, the three factor model for Real Estate PE funds implies that:

$$\begin{aligned}
e_{strip,1,h} &= B^\$_h \\
e_{strip,2,h} &= e_{divm} + e_\pi + B^m_h \\
e_{strip,3,h} &= e_{divreit} + e_\pi + B^{reit}_h
\end{aligned}$$

As another example, the three-factor model for VC funds implies:

$$\begin{aligned}
e_{strip,1,h} &= B^\$_h \\
e_{strip,2,h} &= e_{divsmall} + e_\pi + B^{small}_h \\
e_{strip,3,h} &= e_{divgrowth} + e_\pi + B^{growth}_h
\end{aligned}$$

Similar expressions are obtained for the other fund categories and for models with fewer or more risk factors.

B Point Estimates Baseline Model

B.1 VAR Estimation

In the first stage we estimate the VAR companion matrix by OLS, equation by equation. We start from an initial VAR where all elements of Ψ are non-zero. We zero out the elements whose t-statistic is less than 1.96. We then re-estimate Ψ and zero out the elements whose t-statistic is less than 1.96. We continue this procedure until the Ψ matrix no longer changes and all remaining elements have t-statistic greater than 1.96. The resulting VAR companion matrix estimate, $\widehat{\Psi}$, is listed below.

0.90	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.21	0.00	0.00	0.02	0.00	0.01	0.00	-0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.07	0.00	0.95	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.83	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
-4.77	0.00	0.00	0.00	0.96	0.00	-0.07	0.00	0.00	0.65	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00	0.00	0.37	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	-3.66	0.00	0.00	0.00	0.89	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	1.97	0.00	0.07	0.00	0.09	0.00	-0.07	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00	0.08	0.00	0.00	0.00	0.92	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00	-0.05	0.00	0.00	0.00	0.05	0.00	0.00	0.00	0.00	0.00	0.00	0.00
-9.33	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.87	0.00	0.00	0.00	0.00	0.00
4.62	1.36	-1.72	-7.44	0.05	0.36	0.03	-0.07	-0.06	-0.32	0.04	0.00	0.00	0.00	0.00	0.00
-4.34	0.00	0.00	0.00	0.00	0.00	-0.05	0.00	0.00	0.65	0.00	0.00	0.93	0.00	0.00	0.00
-0.18	0.20	-1.89	-1.84	-0.04	0.12	0.02	-0.19	-0.05	-0.13	-0.01	0.01	0.09	0.11	0.00	0.00
0.00	0.00	0.00	0.00	0.31	0.00	0.00	0.00	-0.15	0.00	0.00	0.00	-0.26	0.00	0.83	-0.36
-1.36	-0.22	0.43	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.05	0.49

The Cholesky decomposition of the residual variance-covariance matrix, $\Sigma^{\frac{1}{2}}$, multiplied by 100 for readability is given by:

0.25	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.04	0.67	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.03	0.06	0.17	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
-0.01	-0.01	-0.08	0.09	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
-1.42	1.11	-1.22	-0.61	8.08	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
-0.03	-0.07	0.11	-0.14	-0.17	2.13	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
-0.82	0.91	-1.17	-0.59	5.42	0.80	7.46	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.29	-0.06	0.04	-0.15	0.03	0.62	-1.51	3.21	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
-0.58	0.70	-0.93	-0.76	6.39	0.97	0.34	0.51	4.45	0.00	0.00	0.00	0.00	0.00	0.00	0.00
-0.08	0.09	0.07	-0.10	-0.42	0.52	-0.08	-0.07	-0.59	1.89	0.00	0.00	0.00	0.00	0.00	0.00
-2.31	1.64	-1.69	-0.10	10.10	-0.06	0.77	0.89	0.10	-0.07	8.57	0.00	0.00	0.00	0.00	0.00
0.33	-0.08	-0.02	0.09	-0.83	2.58	0.52	0.48	-0.69	0.69	-4.37	4.85	0.00	0.00	0.00	0.00
-1.69	1.37	-1.15	-1.13	8.18	-1.15	-1.53	-0.69	-1.16	0.64	0.20	-0.31	4.38	0.00	0.00	0.00
0.08	-0.20	-0.04	-0.13	-0.18	3.14	0.10	0.32	0.36	-0.39	-0.07	0.22	-2.55	2.40	0.00	0.00
-0.14	0.38	-0.26	-0.94	3.48	0.65	0.35	1.23	4.49	0.17	0.24	0.63	0.21	-0.15	6.68	0.00
-0.05	0.08	-0.12	-0.17	0.01	-0.24	0.51	-0.63	-0.74	0.74	-0.26	-0.18	-0.38	0.20	-1.70	3.46

The diagonal elements report the standard deviation of the VAR innovations, each one orthogonalized to the shocks that precede it in the VAR, expressed in percent per quarter.

B.2 Market Price of Risk Estimates

The market prices of risk are pinned down by the moments discussed in the main text. Here we report and discuss the point estimates. Note that the prices of risk are associated with the orthogonal VAR innovations $\varepsilon \sim \mathcal{N}(0, I)$. Therefore, their magnitudes can be interpreted as (quarterly) Sharpe ratios. The constant in the market price of risk estimate $\widehat{\Lambda_0}$ is:

-0.23	0.53	-0.44	-0.05	-0.18	0.68	0.00	0.35	0.00	0.34	0.00	-0.12	0.00	0.03	0.00	0.46

$'$

The matrix that governs the time variation in the market price of risk is estimated to be $\widehat{\Lambda_1} =$:

61.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	11.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	-54.2	-249.7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
54.3	-7.2	0.0	0.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
-6.0	4.0	5.9	9.2	-0.2	4.6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
-104.1	-17.4	-89.1	-161.5	-1.0	-0.0	-1.9	0.0	0.0	7.7	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
100.8	-3.6	-70.5	-31.2	2.3	-7.5	-0.4	0.0	-1.8	-3.3	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
161.6	-6.0	-6.0	-7.0	1.6	-11.1	1.2	0.0	-1.5	-5.3	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
-37.6	29.5	-17.6	-139.5	1.0	-0.0	1.5	0.0	-0.4	-10.3	-2.0	0.4	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
17.1	11.9	-100.0	-105.5	-1.2	0.0	-0.0	-7.0	-1.4	1.0	-0.6	0.5	1.4	0.9	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
-34.1	-13.5	-0.8	9.3	4.0	-0.3	0.1	1.0	-1.8	2.3	-0.1	-0.1	-3.1	1.1	-0.9	-2.6

The first four elements of Λ_0 and the first four rows of Λ_1 govern the dynamics of bond yields and bond returns. The price of inflation risk is allowed to move with the inflation rate. The estimation shows that the price of inflation risk is negative on average ($\widehat{\Lambda_0}(1)$=-0.23), indicating that high inflation states are bad states of the world. The market price of inflation risk becomes larger (less negative) when inflation is higher than average ($\widehat{\Lambda_1}(1,1)$=61.08). The price of real GDP growth risk is positive ($\widehat{\Lambda_0}(2)$=0.53), indicating that high growth states are good states of the world. The price of growth risk increases when GDP growth is above average ($\widehat{\Lambda_1}(2,2)$=10.97). The price of level risk (the shock to short rates that is orthogonal to inflation and real GDP growth) is estimated to be negative ($\widehat{\Lambda_0}(3)$=-0.44), consistent with e.g., the Cox, Ingersoll, and Ross (1985) model. The price of level risk is allowed to change with both the level of interest rates, as in those simple term structure models, and also with the slope factor to capture the fact that bond excess returns are predictable by the slope of the yield curve (Campbell and Shiller, 1991). When interest rate levels are unusually high and the term structure steepens, the price of level risk becomes more negative ($\widehat{\Lambda_1}(3,3)$=-54.26and $\widehat{\Lambda_1}(3,4)$=-249.72), and expected future bond returns increase. The positive association between the slope and future bond returns is consistent with the bond return predictability evidence (Cochrane and Piazzesi, 2006). The price of (orthogonal) slope risk is estimated to be slightly negative on average ($\widehat{\Lambda_0}(4)$=-0.05). Since the spread between the five-year bond yield and the short rate is the fourth element of the state vector, and the short rate is the third element of the state vector, the five year bond yield can be written as:

$$y^{\$}_{t,20} = y^{\$}_{0,20} + (e_{yn} + e_{yspr})' z_t = -\frac{A^{\$}_{20}}{20} - \frac{B^{\$\prime}_{20}}{20} z_t$$

A necessary and sufficient condition to match the five-year bond yield dynamics is to allow for the first four elements of the fourth row of Λ_1 to be non-zero.

The last eight elements of Λ_0 and last eight rows of Λ_1 govern stock pricing. We assume that the market prices of risk associated with the price-dividend ratios are zero, since those variables only play a role as predictors. The only exception is the price-dividend ratio on the stock market. The evidence from dividend strip spot and futures prices and the evidence on strip future returns helps us identify the market prices of risk associated with the pd ratio (fifth element of Λ_t).

The risk prices in the 6th, 8th, 10th, 12th, and 14th rows of Λ_t are chosen to match the observed mean and dynamics of the equity risk premium in model, as shown in Appendix A, and data, as implied by the VAR. We only free up those elements of the 6th, 8th, 10th, 12th, and 14th rows of Λ_1 that are strictly necessary to allow the equity risk premia in the model to move with the same state variables as they do in the VAR. These rows of of Λ_t are also influenced by our insistence on matching the entire time series of the price-dividend ratio on the stock market, real estate, infrastructure, small, and growth stocks.

C Shock-exposure and Shock-price Elasticities

Borovička and Hansen (2014) provide a *dynamic value decomposition*, the asset pricing counterparts to impulse response functions, which let a researcher study how a shock to an asset's cash-flow today affects future cash-flow dynamics as well as the prices of risk that pertain to these future cash-flows. What results is a set of *shock-exposure elasticities* that measure the quantities of risk resulting from an initial impulse at various investment horizons, and a set of *shock-price elasticities* that measure how much the investor needs to be compensated currently for each unit of future risk exposure at those various investment horizons. We now apply their analysis to our VAR setting.

C.1 Derivation

Recall that the underlying state vector dynamics are described by:

$$z_{t+1} = \Psi z_t + \Sigma^{\frac{1}{2}} \varepsilon_{t+1}$$

The log cash-flow growth rates on stocks, REITs, and infrastructure stocks are described implicitly by the VAR since it contains both log returns and log price-dividend ratios for each of these assets. The log real dividend growth rate on an asset $i \in \{m, reit, infra\}$ is given by:

$$\log(D^i_{t+1}) - \log(D^i_t) = \Delta d^i_{t+1} = A^i_0 + A^i_1 z_t + A^i_2 \varepsilon_{t+1},$$

where $A^i_0 = \mu_m$, $A_1 = e'_{divi}\Psi$, and $A^i_2 = e'_{divi}\Sigma^{\frac{1}{2}}$.

Denote the cash-flow process $Y_t = D_t$. Its increments in logs can we written as:

$$y_{t+1} - y_t = \Gamma_0 + \Gamma_1 z_t + z'_t \Gamma_3 z_t + \Psi_0 \varepsilon_{t+1} + z'_t \Psi_1 \varepsilon_{t+1} \tag{27}$$

with coefficients $\Gamma_0 = A^i_0$, $\Gamma_1 = A^i_1$, $\Gamma_3 = 0$, $\Psi_0 = A^i_2$, and $\Psi_1 = 0$.

The one-period log real SDF, which is the log change in the real pricing kernel S_t, is a quadratic function of the state:

$$\log(S_{t+1}) - \log(S_t) = m_{t+1} = B_0 + B_1 z_t + B_2 \varepsilon_{t+1} + z'_t B_3 z_t + z'_t B_4 \varepsilon_{t+1}$$

where $B_0 = -y_0^{\$}(1) + \pi_0 - \frac{1}{2}\Lambda_0'\Lambda_0$, $B_1 = -e_{yn}' + e_\pi'\Psi - \Lambda_0'\Lambda_1$, $B_2 = -\Lambda_0' + e_\pi'\Sigma^{\frac{1}{2}}$, $B_3 = -\frac{1}{2}\Lambda_1'\Lambda_1$, and $B_4 = -\Lambda_1'$.

We are interested in the product $Y_t = S_t D_t$. Its increments in logs can be written as in equation (27), with coefficients $\Gamma_0 = A_0^i + B_0$, $\Gamma_1 = A_1^i + B_1$, $\Gamma_3 = B_3$, $\Psi_0 = A_2^i + B_2$, and $\Psi_1 = B_4$.

Starting from a state $z_0 = z$ at time 0, consider a shock at time 1 to a linear combination of state variables, $\alpha_h'\varepsilon_1$. The shock elasticity $\epsilon(z,t)$ quantifies the date-t impact:

$$\epsilon(z,t) = \alpha_h' \left(I - 2\tilde{\Psi}_{2,t}\right)^{-1} \left(\tilde{\Psi}_{0,t}' + \tilde{\Psi}_{1,t}' z\right)$$

where the $\tilde{\Psi}$ matrices solve the recursions

$$\begin{aligned}
\tilde{\Psi}_{0,j+1} &= \hat{\Gamma}_{1,j}\Sigma^{1/2} + \Psi_0 \\
\tilde{\Psi}_{1,j+1} &= 2\Psi'\hat{\Gamma}_{3,j}\Sigma^{1/2} + \Psi_1 \\
\tilde{\Psi}_{2,j+1} &= \left(\Sigma^{1/2}\right)'\hat{\Gamma}_{3,j}\Sigma^{1/2}
\end{aligned}$$

The $\hat{\Gamma}$ and $\tilde{\Gamma}$ coefficients follow the recursions:

$$\begin{aligned}
\tilde{\Gamma}_{0,j+1} &= \hat{\Gamma}_{0,j} + \Gamma_0 \\
\tilde{\Gamma}_{1,j+1} &= \hat{\Gamma}_{1,j}\Psi + \Gamma_1 \\
\tilde{\Gamma}_{3,j+1} &= \Psi'\hat{\Gamma}_{3,j}\Psi + \Gamma_3 \\
\hat{\Gamma}_{0,j+1} &= \tilde{\Gamma}_{0,j+1} - \frac{1}{2}\log\left(\left|I - 2\tilde{\Psi}_{2,j+1}\right|\right) + \frac{1}{2}\tilde{\Psi}_{0,j+1}\left(I - 2\tilde{\Psi}_{2,j+1}\right)^{-1}\tilde{\Psi}_{0,j+1}' \\
\hat{\Gamma}_{1,j+1} &= \tilde{\Gamma}_{1,j+1} + \tilde{\Psi}_{0,j+1}\left(I - 2\tilde{\Psi}_{2,j+1}\right)^{-1}\tilde{\Psi}_{1,j+1}' \\
\hat{\Gamma}_{3,j+1} &= \tilde{\Gamma}_{3,j+1} + \frac{1}{2}\tilde{\Psi}_{1,j+1}\left(I - 2\tilde{\Psi}_{2,j+1}\right)^{-1}\tilde{\Psi}_{1,j+1}'
\end{aligned}$$

starting from $\hat{\Gamma}_{0,0} = 0$, $\hat{\Gamma}_{1,0} = 0_{1\times N}$, $\hat{\Gamma}_{2,0} = 0_{N\times N}$, and where I is the $N \times N$ identity matrix.

Let $\epsilon_g(z,t)$ be the shock-exposure elasticity (cash-flows $Y = D$) and $\epsilon_{sg}(z,t)$ the shock-value elasticity, then the shock-price elasticity $\epsilon_p(z,t)$ is given by

$$\epsilon_p(z,t) = \epsilon_g(z,t) - \epsilon_{sg}(z,t).$$

In an exponentially affine framework like ours, the shock price elasticity can also directly be derived by setting $Y_t = S_t^{-1}$ or $y_{t+1} - y_t = -m_{t+1}$, with coefficients in equation (27) equal to $\Gamma_0 = -B_0$, $\Gamma_1 = -B_1$, $\Gamma_3 = -B_3$, $\Psi_0 = -B_2$, and $\Psi_1 = -B_4$.

The shock-price elasticity quantifies implied market compensation for horizon-specific risk exposures. In our case, these risk compensations are extracted from a rich menu of observed asset prices matched by a reduced form model, rather than by constructing a structural asset pricing model. The horizon-dependent risk prices are the multi-period impulse responses for the cumulative stochastic discount factor process.

C.2 Results

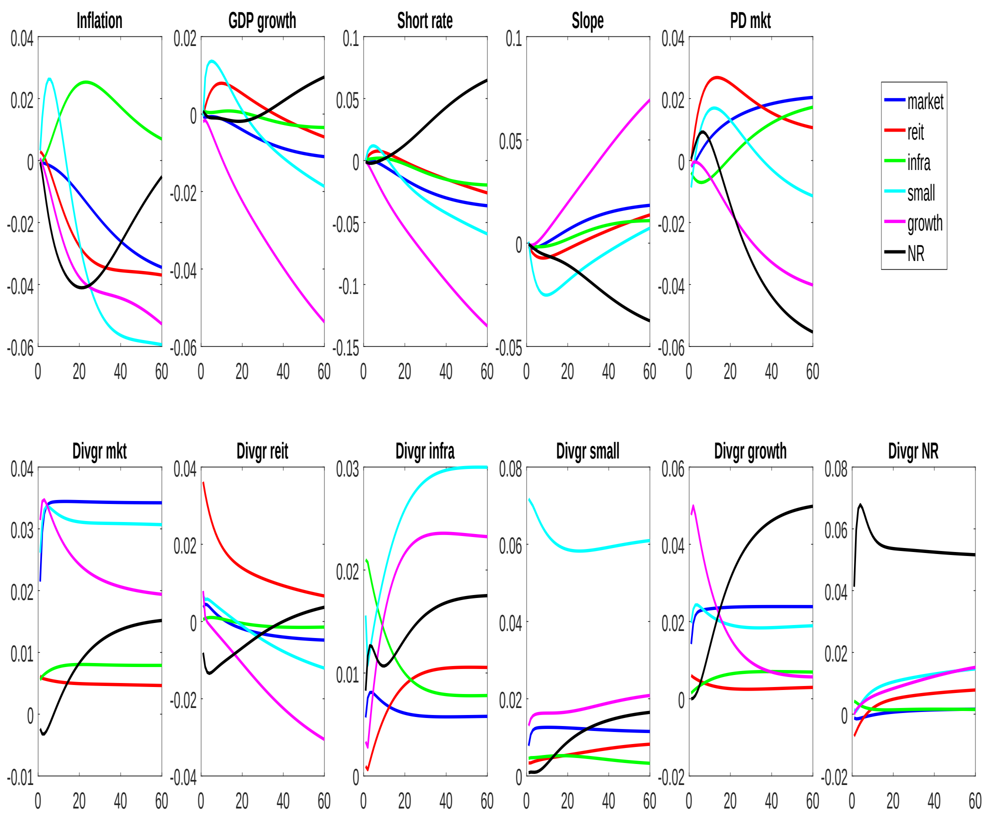
Figure C.14 plots the shock-exposure elasticities of the six dividend growth processes -on the market (blue), REITs (red), infrastructure (green), small (cyan), growth (magenta), and natural resource stocks (black)- to a one-standard deviation shock to inflation (top left), real per capital GDP growth (top second), the short rate (top third), the slope factor (top fourth), , the price-dividend ratio on the market (top right), the dividend growth rate on the market (bottom left), the dividend growth rate on REITs (bottom middle), the dividend growth rate on infrastructure (bottom middle), the dividend growth rate on small stocks (bottom fourth), and the dividend growth rate on growth stocks (bottom right). The shock exposure elasticities are essentially impulse-responses to the original (i.e., non-orthogonalized) VAR innovations. They describe properties of the VAR, not of the asset pricing model. Since our private equity cash-flows are linear combinations of these dividends, the PE cash flow exposures to the VAR shocks will be linear combinations of the plotted shock exposure elasticities of these three dividend growth rates.

There is interesting heterogeneity in the cash flow exposures of the five risky assets to the VAR shocks. For example, the top left panel shows that infrastructure and to a lesser extent small stock cash flows increase in the wake of a positive inflation shock, while the dividend growth responses for the aggregate stock market and especially for REITs, growth stocks, and natural resource stocks are negative. This points to the inflation hedging potential of infrastructure assets and the inflation risk exposure of REITs, growth stocks, natural resource stocks, and the market as a whole. The second panel shows that REIT and small stock dividend growth responds positively to a GDP growth shock, while cash flow growth on the market and on growth stocks (ironically) respond negatively. It is well known that the market portfolio as a whole is fairly growth-oriented. NR stocks have strong exposure to GDP growth especially at longer horizons. All cash flows except NR respond negatively to an increase in interest rates in the long-run. The response of REIT and small stock cash flows is positive in the short run. REIT cash flows are rents which can be adjusted upwards when rates increase, which typically occurs in a strong economy (see the GDP panel). Small and growth stocks have a lot more interest rate level risk exposure than the other asset classes, while NR cash flows are a great interest rate hedge. The market dividend growth shows a substantial positive response to a steepening yield curve. Slope exposure is even larger for growth stocks. When the slope steepens, which tends to happen during recessions, NR cash flows fall. The bottom panels show that the dividend growth shock to the market is nearly permanent, while the other five cash flow shocks are mean-reverting. A positive innovation in REIT cash flows is associated with a decline in cash flows on growth firms. This makes sense given the value-like behavior of REITS (Van Nieuwerburgh, 2019). A positive shock to infrastructure cash flows has strong and often long-lasting positive effects on the other five cash flow series (bottom middle panel). Positive growth stock cash flow innovations have strongly persistent positive effects for NR cash flows. NR cash flows have few spillovers on the other stock segments.

Figure C.15 plots the shock-price elasticities to a one-standard deviation shock to each of the same (non-orthogonalized) VAR innovations. Shock price elasticities are properties of the SDF process, and therefore depend on the estimated market price of risk parameters. They quantify the compensation investors demand for horizon-dependent risk exposures. The price of inflation risk is negative, consistent with increases in inflation being bad states of the world. GDP growth risk is naturally priced positively, and more so at longer horizons. Level risk is negatively priced, consistent with standard results in the term structure literature that consider high interest rate periods bad states of the world. The price of level risk

FIGURE C.1: Shock Exposure Elasticities

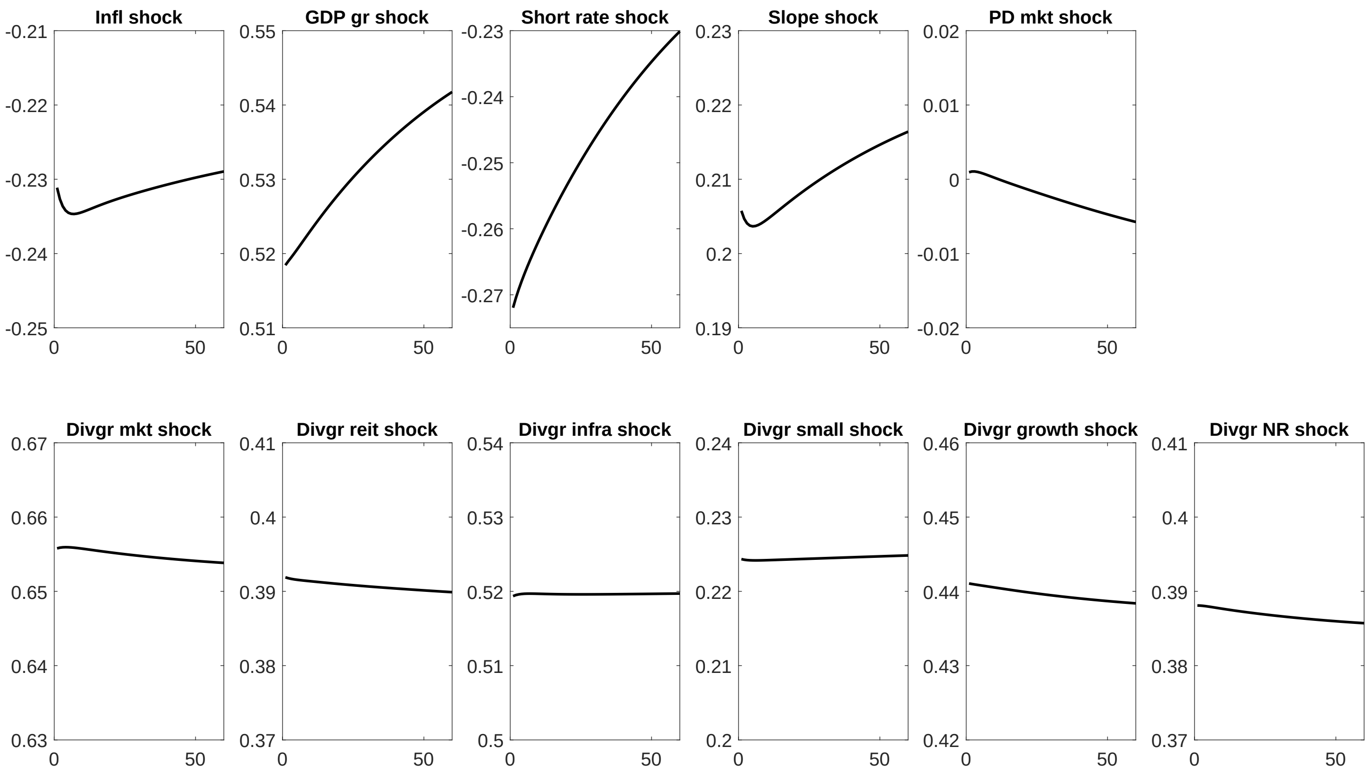
The figure plots the shock-exposure elasticities of dividend growth on the market, dividend growth of REITs, dividend growth of infrastructure shocks, dividend growth of small stocks, dividend growth of growth stocks, and dividend growth of natural resource stocks to a one-standard deviation shock to inflation (top left), real GDP growth (top second), the short rate (top third), the slope factor (top right), the price-dividend ratio on the market (top right), dividend growth rate on the market (bottom left), dividend growth rate on REITs (bottom second), dividend growth rate on infrastructure (bottom third), dividend growth rate on small stocks (bottom fourth), dividend growth on growth stocks (bottom fifth), and dividend growth of natural resource stocks (bottom right). The shock exposures are to the non-orthogonalized VAR innovations $\Sigma^{\frac{1}{2}}\varepsilon$.



becomes less negative at longer horizons. The price of slope risk is positive, consistent with the findings in (Koijen, Lustig, and Van Nieuwerburgh, 2017). All cash-flow shocks in the bottom five panels naturally have positive risk prices since increases in cash-flow growth are good shocks to the representative investor. The highest risk price is associated with shocks to the aggregate stock market, followed by infrastructure shocks, then growth shocks. Compensation for exposure to the various dividend shocks varies little with the horizon.

FIGURE C.2: Shock Price Elasticities

The figure plots the shock-price elasticities to a one-standard deviation shock to the inflation factor (top left), real GDP growth (top second), short rate (top third), the slope factor (top right), the price-dividend ratio on the market (top right), the dividend growth rate on the market (bottom left), the dividend growth rate on REITs (bottom second), the dividend growth rate on infrastructure (bottom third), the dividend growth rate on small stocks (bottom fourth), the dividend growth on growth stocks (bottom fifth), and the dividend growth on natural resources stocks (bottom right). The shocks whose risk prices are plotted are the (non-orthogonalized) VAR innovations $\Sigma^{\frac{1}{2}}\varepsilon$.



D Korteweg-Nagel Details

D.1 Connection between our approach, GPME, and PME

Korteweg and Nagel (2016) define their GPME measure for fund i as:

$$GPME_t^i - 1 = \sum_{h=0}^{H} M_{t+h}^h X_{t+h}^i - 1 \tag{28}$$

$$= \sum_{h=0}^{H} M_{t+h}^h \left\{ \boldsymbol{q}_{t,h}^i \boldsymbol{F}_{t+h} + v_{t+h}^i \right\} - 1$$

$$= RAP_t^i + \mathbb{E}_t \left[\sum_{h=0}^{H} M_{t+h}^h \boldsymbol{q}_{t,h}^i \boldsymbol{F}_{t+h} \right] - 1 + \sum_{h=0}^{H} M_{t+h}^h \boldsymbol{q}_{t,h}^i \boldsymbol{F}_{t+h} - \mathbb{E}_t \left[\sum_{h=0}^{H} M_{t+h}^h \boldsymbol{q}_{t,h}^i \boldsymbol{F}_{t+h} \right]$$

$$= RAP_t^i + \sum_{h=0}^{H} \boldsymbol{q}_{t,h}^i \left(M_{t+h}^h \boldsymbol{F}_{t+h} - \mathbb{E}_t [M_{t+h}^h \boldsymbol{F}_{t+h}] \right) \tag{29}$$

If the SDF model is correct, $\mathbb{E}_t[GPME^i] = 1$. The difficulty with computing (28) is that it contains the *realized* SDF which is highly volatile. In KN's implementation, the SDF is a function of only the market return: $M^h_{t+h} = \exp(0.088h - 2.65 \sum_{k=0}^{h} r^m_{t+k})$. If the realized market return over a 10 year period is 100%, the realized SDF is 0.17. If the stock return is 30%, the SDF is 1.08. Because of the multiple sources of risk and the richer risk price dynamics, our SDF is substantially more volatile than one considered in KN; it has a higher maximum Sharpe ratio. The realizations of the SDF are on average much lower than in KN, so that the GPME approach leads to unrealistically low PE valuations. Our methodology solves this issue because it avoids using the realized SDF and instead relies on strip prices, which are *expectations* of SDFs, multiplied by cash-flows.

A second difference between the two approaches is that the realized GPME can be high (low) because the factor payoffs F_{t+h} are unexpectedly high (low). This is the second term in equation (29). Our measure does not credit the GP for this unexpected, systematic cash-flow component. RAP removes a "factor timing" component of performance that is due to taking risk factor exposure. Like our approach, the simple PME does not credit the GP with factor timing.

Third, our approach credits the GP for "investment timing" skill while the GPME approach does not. Because it assumes that the replicating portfolio deploys the entire capital right away, a manager who successfully waits a few periods to invests will have a positive RAP. If the GP harvests at a more opportune time than the replicating portfolio, whose harvesting timing is determined by the average PE fund in that vintage and category, this also contributes to the RAP. The GPME as well as the simple PME approach do not credit the manager for investment timing because they assume that the replicating portfolio follows the observed sequence of PE capital calls and distributions.

Fourth, our approach accommodates heterogeneity in systematic risk exposure across PE funds that differ by vintage and category. In the standard PME approach, the market beta of each fund is trivially the same and equal to 1. In the simplest implementation of the GPME approach, PE funds are allowed to have a market beta that differs from 1, but the beta is the same for all funds. We allow for multiple risk factors, and the exposures differ for each vintage and for each fund category. Because market prices of risk vary with the state of the economy, so does the RAP. The next section provides more detail on the KN approach and more discussion on the points of differentiation.

D.2 More Details

They propose:

$$m_{t+1} = a - b r^m_{t+1},$$

whereby the coefficients a and b are chosen so that the Euler equation $1 = E[M_{t+1}R_{t+1}]$ holds for the public equity market portfolio and the risk-free asset return. More specifically, they estimate $a = 0.088$ and $b = 2.65$ using a GMM estimator:

$$\min_{a,b} \left(\frac{1}{N} \sum_i u_i(a,b) \right)' W \left(\frac{1}{N} \sum_i u_i(a,b) \right)$$

where

$$u_i(a,b) = \sum_{j=1}^{J} M_{t+h(j)}(a,b)[X_{if,t+h(j)}, X_{im,t+h(j)}],$$

N is the number of funds, and W is a 2×2 identity matrix. The T-bill benchmark fund cash-flow, X_{if}, and the market return benchmark cash-flow, X_{im}, are the cash-flows on a T-bill and stock market investment, respectively, that mimic the timing and magnitude of the private equity fund i's cash-flows. The $t + h(j)$ are the dates on which the private equity fund pays out cash-flow $j = 1, \cdots, J$. Date t is the date of the first cash-flow into the fund, so that $h(1) = 0$. For each of the two benchmark funds, the inflows are identical in size and magnitude as the inflows into the PE fund. If PE fund i makes a payout at $t + h(j)$, the benchmark funds also make a payout. That payout consists of two components. The first component is the return on the benchmark since the last cash-flow date. The second component is a return of principal, according to a preset formula which returns a fraction of the capital which is larger, the longer ago the previous cash-flow was.

A special case of this model is the public market equivalent of Kaplan and Schoar (2005), which sets $a = 0$ and $b = 1$. This is essentially the log utility model. The simple PME model is rejected by Korteweg and Nagel (2016), in favor of their generalized PME model.

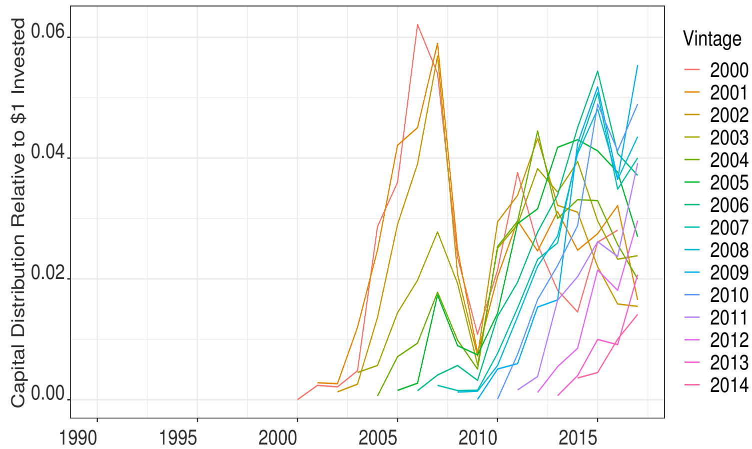
There are several key differences between our method and that of Korteweg and Nagel (2016). First, we do not use SDF realizations to discount fund cash-flows. Rather, we use bond prices and dividend strip prices, which are conditional expectations. Realized SDFs are highly volatile. Second, the KN approach does not take into account heterogeneity in the amount of systematic risk of the funds. All private equity funds are assumed to have a 50-50 allocation to the stock and bond benchmark funds. Our model allows for different funds to have different stock and bond exposure. Third, the KN approach uses a preset capital return policy which is not tailored to the fund in question. For example, a fund may be making a modest distribution in year 5, say 10%, and a large distribution in year 10 (90%). Under the KN assumption, the public market equivalent fund would sell 50% in year 5 and the other 50% in year 10. There clearly is a mismatch between the risk exposure of the public market equivalent fund and that of the private equity fund. In other words, the KN approach does not take into the account the magnitude of the fund distributions, only their timing. Fourth, we use additional risk factors beyond those considered in KN.

To study just the importance of the last assumption, we can redo our calculations using a much simplified state vector that only contains the short rate, inflation, and the stock market return. This model has constant risk premia.

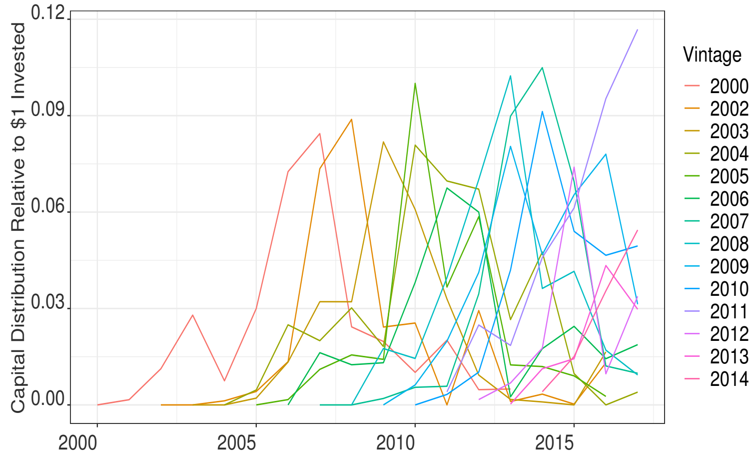
E Additional Results

FIGURE E.1: Cash-Flows by Vintage, Alternate Categories

(a) Fund of Funds



(b) Debt Funds



(c) Restructuring

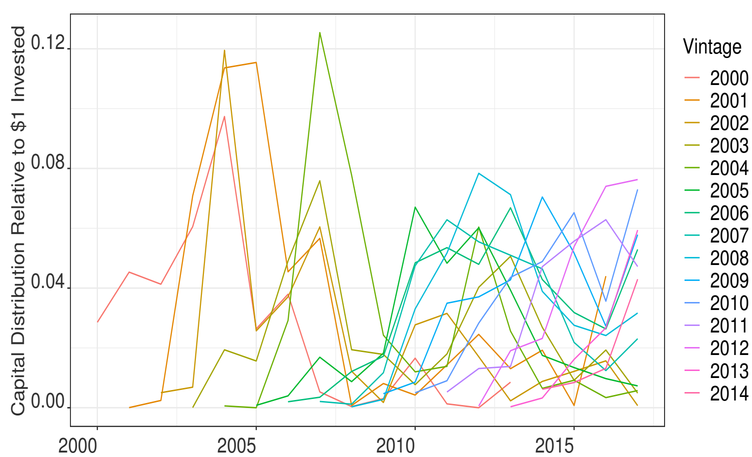
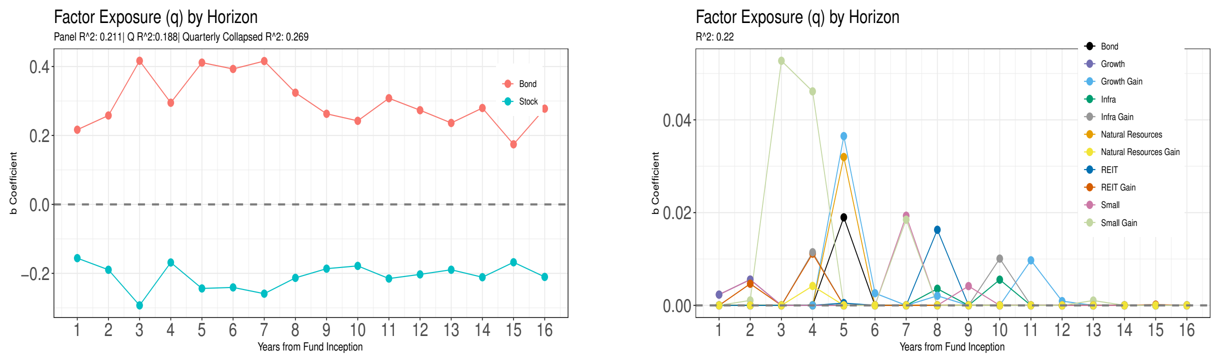
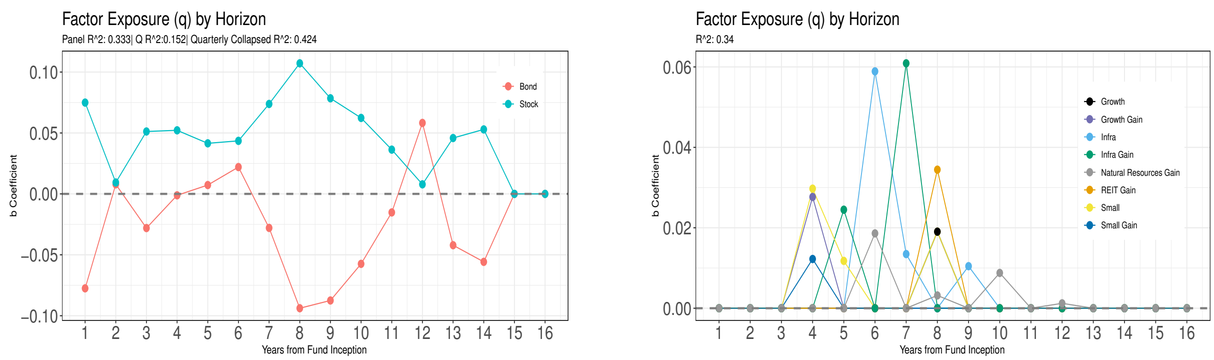


FIGURE E.2: Factor Exposure for other Categories by Fund Horizon

Panel A: Restructuring



Panel B: Debt Fund



Panel C: Fund of Funds

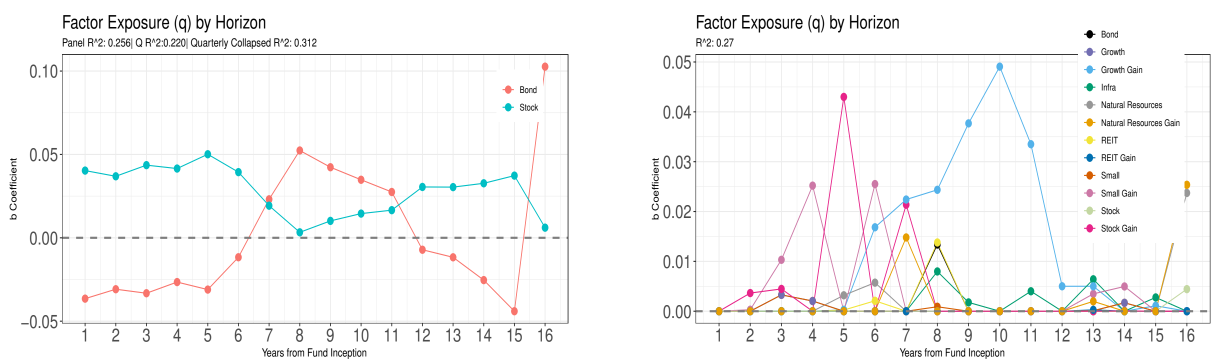
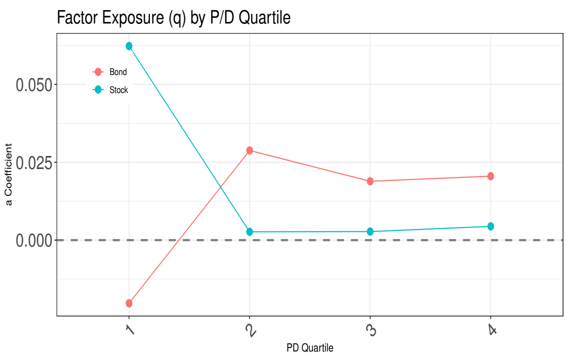
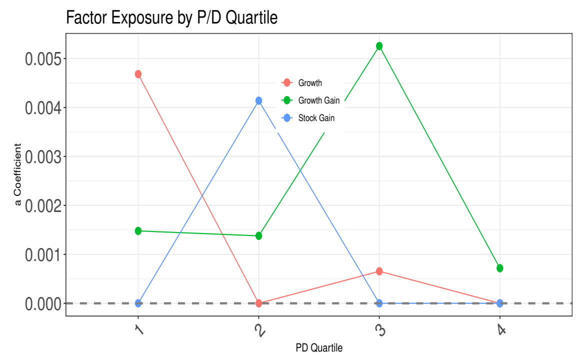


FIGURE E.3: Factor Exposure by P/D Quartile

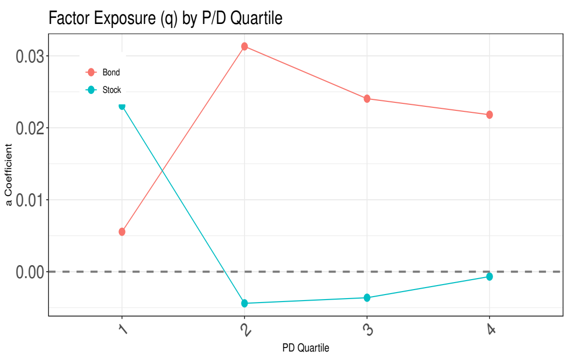
2-Factor | Lasso

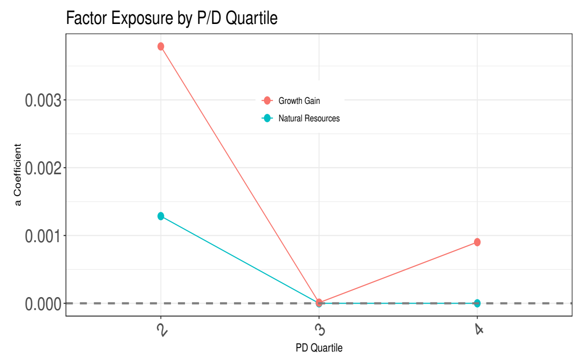
Panel A: Buyout



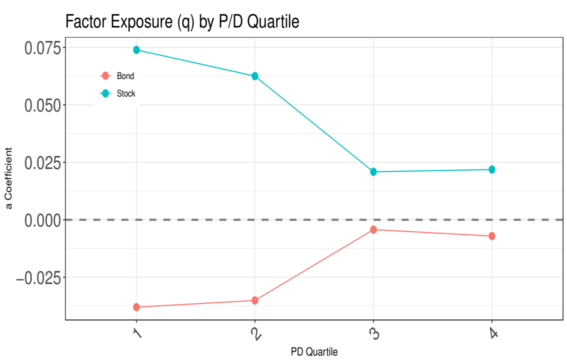


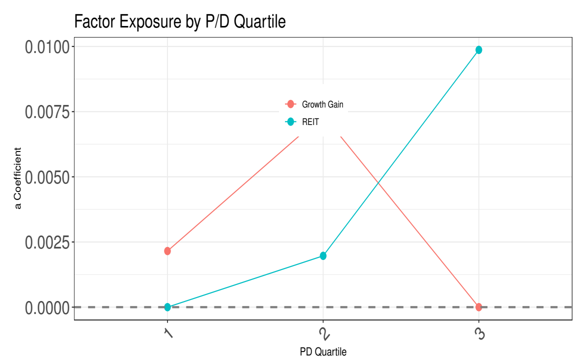
Panel B: Venture Capital



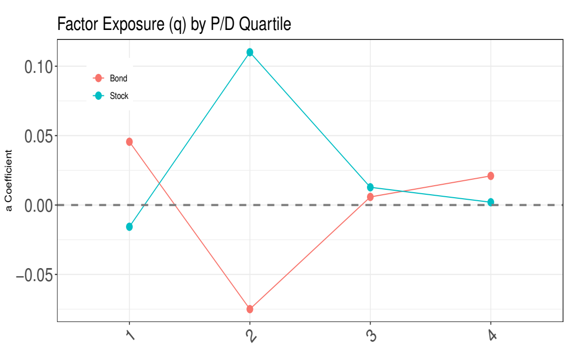


Panel C: Real Estate





Panel D: Infrastructure



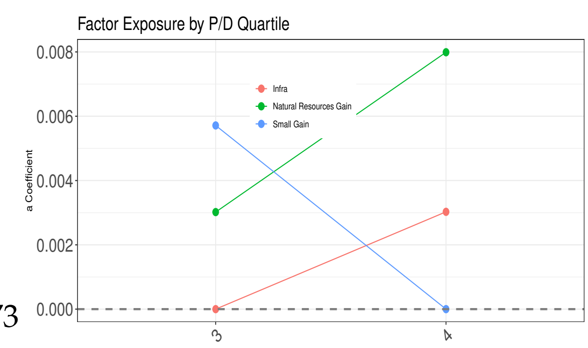
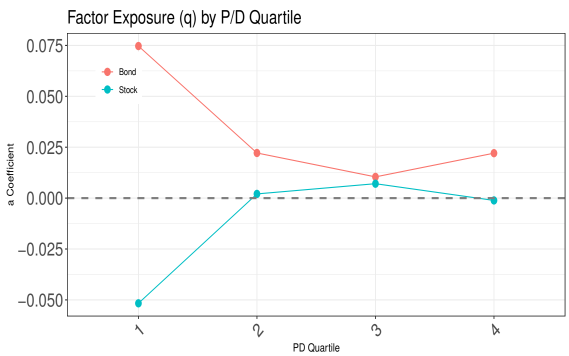
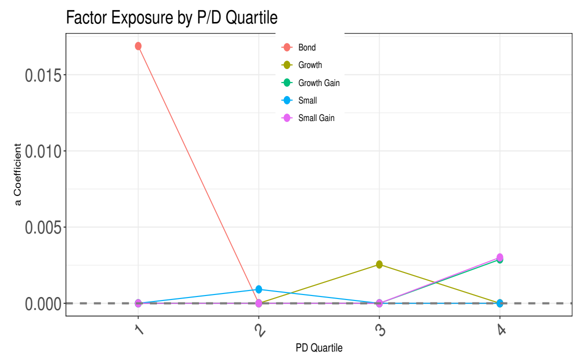


FIGURE E.4: Factor Exposure by P/D Quartile

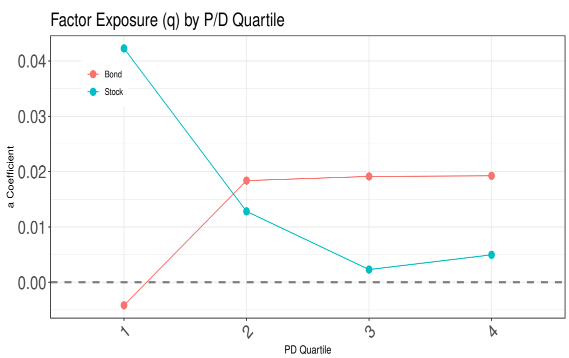
2-Factor Lasso

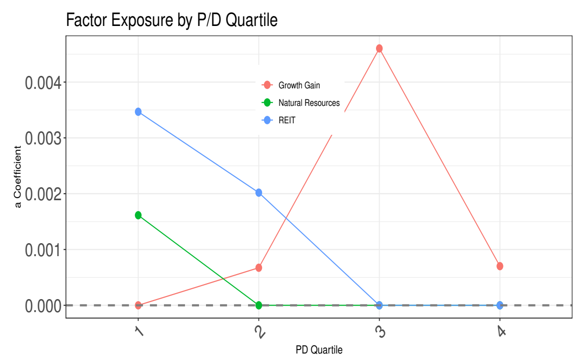
Panel A: Restructuring



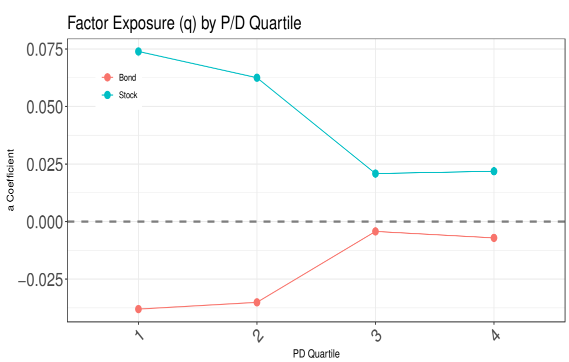


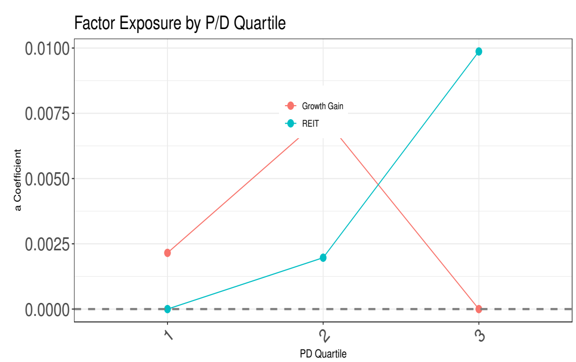
Panel B: Fund of Funds





Panel C: Debt Fund





Panel D: Infrastructure

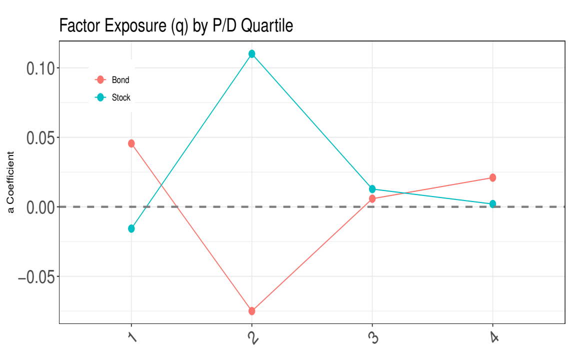
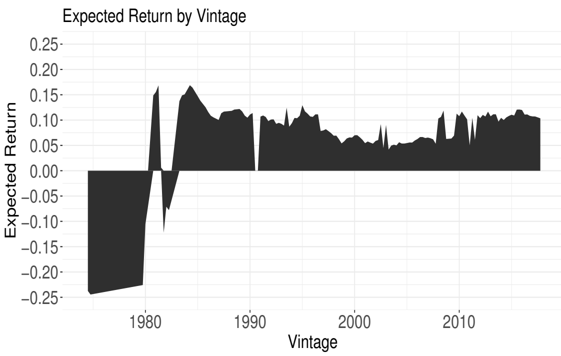
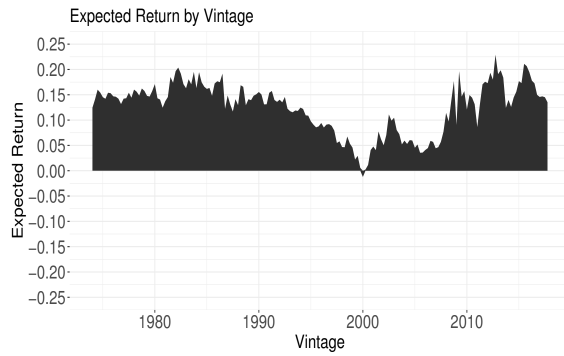


FIGURE E.5: Expected Returns by Vintage for Additional Categories

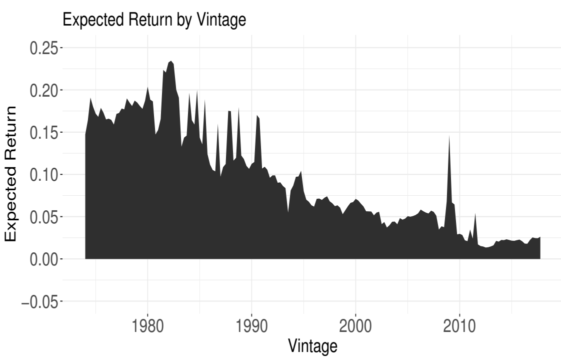
2-Factor | Lasso

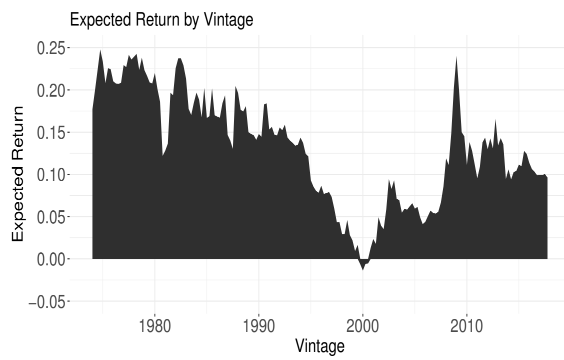
Panel A: Restructuring



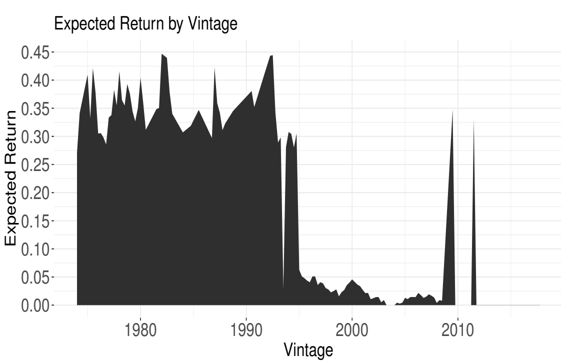


Panel B: Fund of Funds





Panel C: Debt Fund



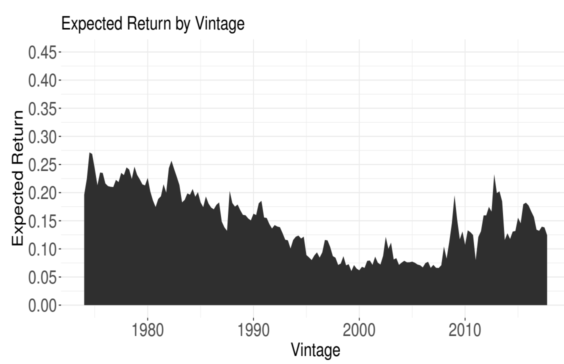
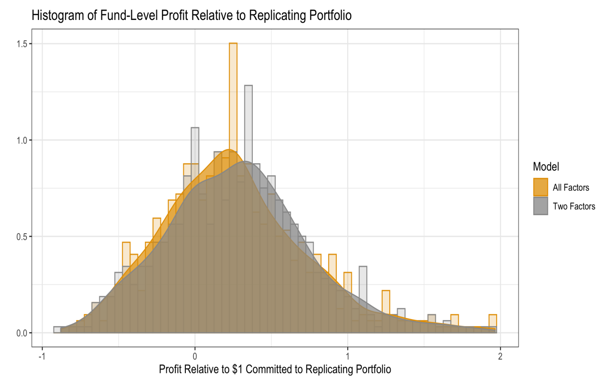


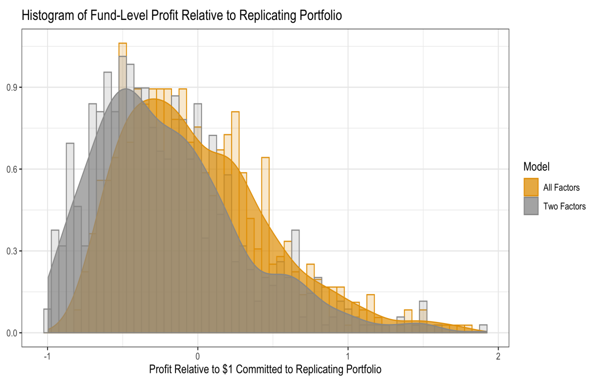
FIGURE E.6: Profit Comparison for Alternate Categories

2-Factor | Lasso

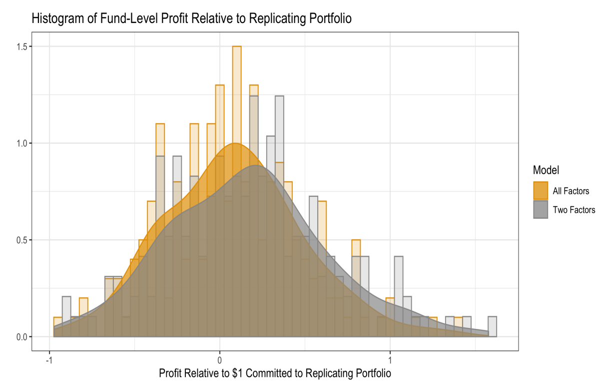
Panel A: Buyout



Panel B: Venture Capital



Panel C: Real Estate



Panel D: Infrastructure

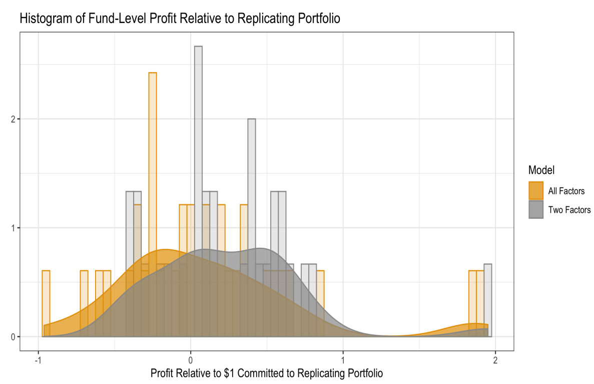
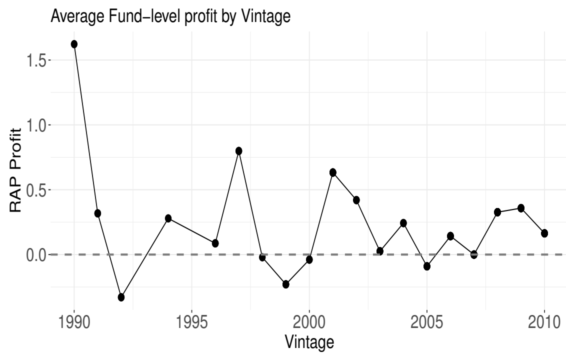
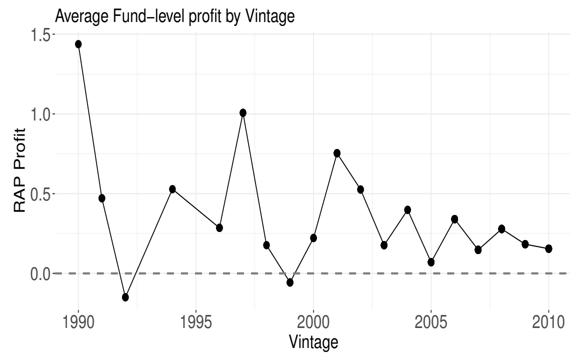


FIGURE E.7: Profits Over Time for Additional Categories

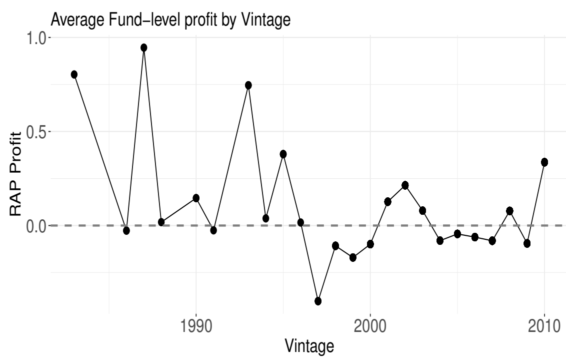
2-Factor | Lasso

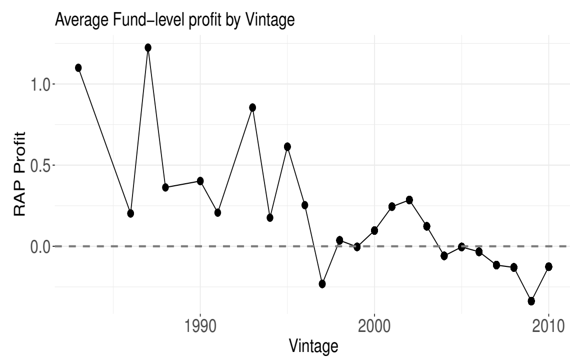
Panel A: Restructuring



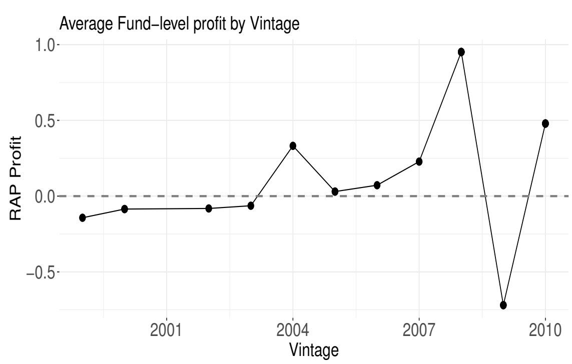


Panel B: Fund of Funds





Panel C: Debt Fund



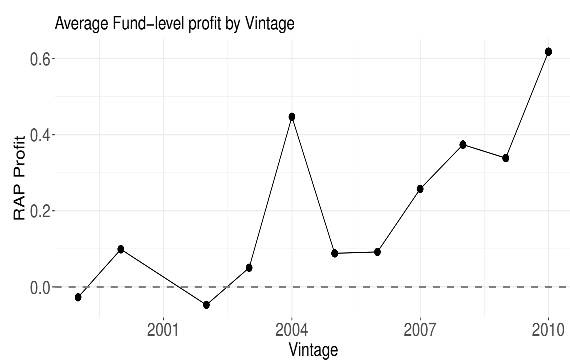
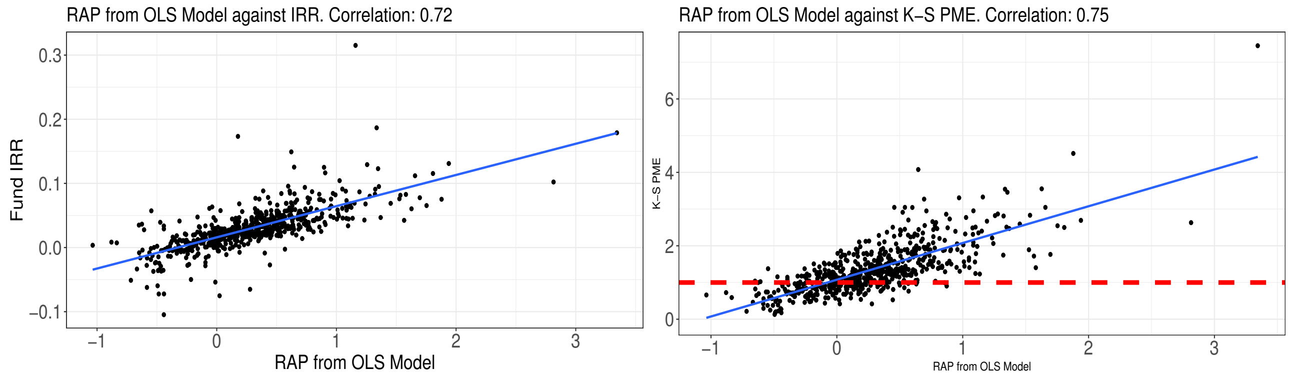
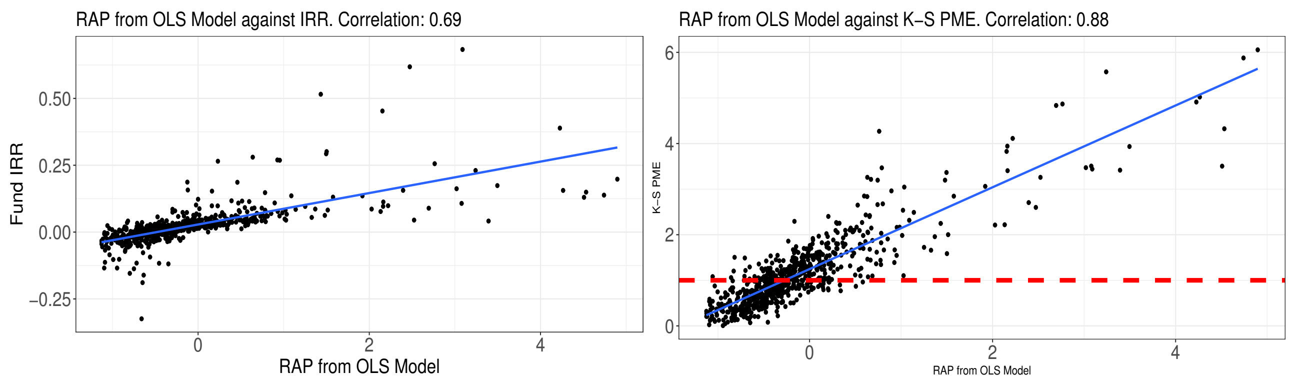


FIGURE E.8: OLS Model Alternative Approach Comparison

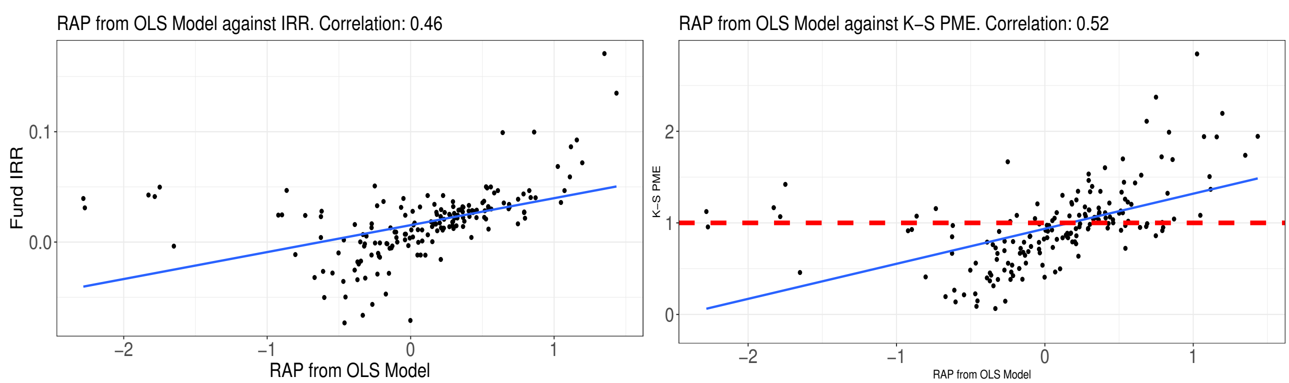
Panel A: Buyout



Panel B: Venture Capital



Panel C: Real Estate



Panel D: Infrastructure