

Extrapolation Bias and Dynamic Liquidity Management*

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Abstract

We consider the optimal dynamic liquidity management of a financially constrained firm when its manager is risk-neutral but ambiguity-averse with respect to the firm's future cash flows. Managerial ambiguity-aversion generates endogenous time-varying worst-case beliefs that overweight recent cash flow realizations, thereby providing a microeconomic foundation for managerial extrapolation bias. Moreover, managerial ambiguity-aversion has different implications on firms' liquidity management and recapitalization policies than risk. Models with risk alone imply that higher cash flow volatility increases firms' payout and refinancing thresholds. By contrast, our model predicts that when ambiguity-averse managers face a higher long-term cash flow uncertainty, they optimally reduce firms' payout and refinancing thresholds. Implications for investment and implied ambiguity premia are also studied.

Keywords: Ambiguity-aversion, Extrapolation bias, Liquidity management, Investment, Risk management, Permanent and Temporary shocks

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Uncertainty is one of the fundamental facts of life. It is as ineradicable from business decisions as from those in any other field. —Frank H. Knight (1921), Part III, Ch. XII.

1 Introduction

Optimal investment and liquidity planning are central to the practice of corporate finance. Managers exhibit precautionary concerns if in the future they may be deprived of the funds that will enable them to take advantage of growth opportunities, execute optimal investment, or just stay alive. Moreover, the difficulty in understanding the factors that affect future profits generate cash flow uncertainty. Therefore, managers confront uncertainty in the sense of Knight (1921) and are unable to assign unique probabilities to future outcomes.¹ Indeed, according to Knight, profit is a reward to an entrepreneur for bearing uncertainty.

As the Ellsberg paradox (1961) demonstrates, individuals are averse to uncertainty, preferring gambles with known probabilities to those with unknown probabilities. Gilboa and Schmeidler (1989) resolve the Ellsberg paradox by postulating a multiple-priors model. A decision-maker does not have enough information to form a prior belief; instead, she maintains a set of prior distributions and believes that any one of them may be the true prior. Furthermore, she is averse to this ambiguity and evaluates a gamble according to the minimal expected utility over all priors in the set. Chen and Epstein (2002) extend the multiple-priors model to a dynamic setting using a recursive approach that is easily implemented via standard dynamic programming techniques.²

The purpose of this paper is to provide a first step towards understanding the impact of ambiguity-aversion on liquidity management, investment decisions, and belief formation in a dynamic setting. To do so, we develop a model of dynamic investment and liquidity management of firms following Bolton et al. (2013). Our model departs from theirs in two dimensions. First, we model liquidity constrained firms subject to permanent (growth) shocks and temporary (cash flow) shocks. As argued by Décamps et al. (2017), firms' cash flows cannot be described adequately using either temporary or permanent shocks alone. Production and demand shocks are of a temporary nature and unlikely to affect long-term prospects. By contrast, long-term cash flows also change

¹By contrast, outcomes with known probabilities are termed “risky” rather than “uncertain.”

²They show that an optimal recursive value function is dynamically consistent when the decision-maker's set of priors satisfies a rectangularity condition. For a more thorough analysis of the role played by the rectangularity condition in the recursivity of utility and time-consistency see Epstein and Schneider (2003).

over time due to industry or macroeconomic shocks of a permanent nature. Second, we incorporate an ambiguity-averse manager who is unsure about the true distributions of these shocks.³ To model managers' ambiguity-aversion, we adopt the Chen and Epstein (2002) framework. This approach embeds the Gilboa and Schmeidler (1989) max-min approach to ambiguity-aversion in a continuous time recursive setting. An alternative would be to model the degree of ambiguity aversion using the smooth ambiguity-aversion approach of Klibanoff et al. (2005) or the multiplier preferences developed by Anderson, Hansen, and Sargent (2003).⁴ We find the Chen and Epstein approach more tractable in our setting.⁵

Our two-shock environment allows us to draw richer implications in comparison to models using temporary shocks alone. We model two stochastic processes: the firm's size and its contemporaneous profitability. The product of these two processes gives firm's cash flows. Shocks to firm size have a permanent effect on a firm's cash flows, whereas shocks to firm profitability have only a temporary effect on its cash flows.

A critical feature in our model is that the manager is ambiguity-averse to *both* shocks, even though he is risk neutral. He is unsure about the true model (i.e., distribution) generating the temporary and permanent shocks. Instead, he devises a reference model from his best guess, but he considers a set of plausible models around the reference model and displays aversion toward this model uncertainty (i.e., ambiguity).

The timing of the model is as follows: an all-equity firm starts with a given amount of cash and capital. These state variables fluctuate with the trajectories of shocks and the optimal investment and financing policies. Once it has sufficient cash reserves, a manager optimally distributes dividends. When it runs out of cash, the firm either gets liquidated or issues new equity, depending on the cost of external finance. Upon liquidation, the existing shareholders collect the recovery value of assets net of the deadweight costs of liquidation.

Our main result states that managerial ambiguity-aversion generates *endogenous* worst-case

³We assume no agency frictions between a firm's shareholders and its manager; therefore we assume that the manager and the shareholders are ambiguity-averse and display the same degree of aversion. For models featuring shareholder-manager conflicts in the presence of permanent and temporary shocks see Hackbarth et al. (2018) and Gryglewicz et al. (2018).

⁴See Miao and Ju (2012) for an application of the smooth ambiguity model in asset pricing, Miao and Rivera (2016) for an application of multiplier preferences in a dynamic principal-agent setting, and Strzalecki (2011) for axiomatic foundations of multiplier preferences.

⁵See Epstein and Schneider (2010) for applications of the recursive max-min model in asset pricing and macroeconomics.

time-varying beliefs that overweight recent realizations. In other words, the manager in our model behaves “as if” he displays extrapolation bias. The intuition is simple: when the marginal value of liquidity is high, the manager is effectively more pessimistic about cash flow (temporary) shocks and less concerned about growth (permanent) shocks. The manager’s marginal value of liquidity grows after the firm experiences a series of negative cash flow shocks. As a consequence, the manager becomes relatively more averse to model uncertainty with respect to the firm’s future cash flows than to the long-term growth prospects. By contrast, when the firm starts to generate positive cash flows and accumulates enough cash reserves, the threat of liquidation becomes less of a concern; therefore, the marginal value of liquidity decreases. Hence, the ambiguity-averse manager becomes less concerned (i.e., more optimistic) about future cash flows. That is, the ambiguity-averse manager behaves *as if* he overweights recent cash flow realizations when forming his expectations about the future. To this extent, we argue that the interaction between a manager’s optimal allocation of worst-case beliefs and financial frictions provides *a* microeconomic foundation of managerial extrapolation bias.⁶

It is fair to compare our model with an alternative formulation with exogenous extrapolation bias. One can ask how our model differs from one with directly hard-wired extrapolation bias. In particular, one could consider a model in which the manager forms his expectation by imposing relatively higher weights on more recent observations.⁷ However, to the best of our knowledge, our model is simpler than models with exogenous extrapolation bias, which require modeling an explicit learning process. More importantly, our model provides a microeconomic foundation of such bias. In our model, extrapolation bias arises endogenously from the manager’s optimization problem when he faces model uncertainty about the distribution of the firm’s shocks.

There are a number of additional results. First, managers’ ambiguity-aversion toward permanent and temporary shocks have different implications for firms’ cash policies. When an ambiguity-averse manager faces a higher level of uncertainty about permanent (temporary) shocks, he optimally reduces (increases) the firm’s payout and refinancing thresholds. Intuitively, higher ambiguity

⁶Excessive extrapolation has been empirically documented in the portfolio choices of individual investors (e.g., Chevalier and Ellison, 1997; Sirri and Tufano, 1998), in the residential housing market (e.g., Shiller, 2005), in laboratory experiments (e.g., De Bondt, 1993), in the formation of inflation expectations (e.g., Malmendier and Nagel, 2015), in analysts forecast of future earnings (Hoberg and Phillips, 2010), in corporate investment decisions (Paaso, 2018), and in investors demand for structured retail products (Shin 2018).

⁷We conjecture that we could apply “recency biased learning” as in Bansal and Shaliastovich (2010).

about temporary shocks makes the manager pessimistic about the firm's ability to generate enough internal cash flows to avoid liquidation. Thus, it is optimal to delay payout. By contrast, higher ambiguity about permanent shocks makes the manager pessimistic about the growth rate of the firm. Hence, less cash is needed to protect a smaller expected future asset base. Importantly, this asymmetry is unique to the ambiguity-aversion model when compared to the model with only risk. Because higher volatility of either the temporary or permanent shocks necessarily imply higher precautionary levels of cash, the firm's dividend and refinancing thresholds also increase when volatilities increase.

Second, ambiguity always reduces the firm's investment rate in the absence of financial frictions. However, in the presence of financial frictions, ambiguity can increase investment for low levels of cash reserves and decrease investment for high levels of cash reserves. Optimal investment trades off the benefit of investment versus the cost of liquidity. Ambiguity reduces the benefit of investment, but it can reduce the cost of liquidity. For high levels of cash, the first effect dominates, reducing investment. However, for low levels of cash the second effect dominates, increasing investment. Critically, we show that the effect of ambiguity on investment depends on whether the outside investors purchasing the firm after liquidation are ambiguity-averse or not.

Finally, we compute the ambiguity premia generated by the model. In particular, we compute the excess return an ambiguity-neutral investor would expect from holding the firm's stock. We find that the ambiguity premia associated with temporary shocks is increasing in the firm's level of financial distress. Surprisingly, the ambiguity premia associated with permanent shocks is hump-shaped in the level of financial distress. Thus, our model provides a resolution for the distress risk puzzle documented by Campbell, Hilscher, and Szilagyi (2008), stating that financially distressed firms have low expected returns. Moreover, according to the model, the distress risk puzzle should be concentrated in firms with high ambiguity about permanent shocks and for whom liquidation is particularly inefficient. We argue this is consistent with evidence that the distress risk puzzle is stronger for firms with larger asset bases, low R&D spending, and high industry sales concentration as documented by Garlappi and Yang (2011).

Our paper is related to a fast growing literature on dynamic corporate finance with liquidity frictions.⁸ These recent advances in dynamic investment models enrich our understanding of how

⁸See Moreno-Bromberg and Rochet (2018) for a recent survey.

external financial frictions influence firms' optimal investment. Bolton et al. (2011) show that the endogenous marginal value of liquidity plays a vital role in determining optimal investment and liquidity. Décamps et al. (2017) show that a model with both permanent and temporary shocks delivers a more realistic behavior in terms of payout, financing, and cash flow to cash sensitivity. Our paper is the first to consider not only risk, but also ambiguity with respect to these shocks.

Our paper also adds to the recent literature on applications of ambiguity-aversion in corporate finance settings. In the model of Dicks and Fulghieri (2015), ambiguity-aversion leads to endogenous disagreement between firm insiders and external shareholders, thus creating a motive for governance. Furthermore, ambiguity-aversion generates strategic complementarities between investments in innovative projects, rationalizing innovation and merger waves (Dicks and Fulghieri, 2016). Lee (2015) derives the optimal capital structure of a firm when its manager is ambiguity-averse. His model predicts lower leverage than traditional trade-off models based on risk alone. Breuer et al. (2017) develop a static model in which investors are ambiguity-averse with respect to the value of future investments. Consistent with our results, firms do not want to reserve cash to finance future investments because the ambiguity-averse investors place excessive weight on the pessimistic distribution over future investment values. Malenko and Tsoy (2018) study optimal security design under ambiguity-aversion and find that standard outside equity and standard risky debt arise as equilibrium securities. Finally, Garlappi et al. (2012) address the effect of ambiguity on real investment with expansion and contraction options. In particular, they show that an ambiguity-averse manager is reluctant to invest but also reluctant to abandon projects (escalating commitment). Moreover, the anticipation of future reluctance to abandon may induce ambiguity-averse agents to forego investment in the first place.⁹ Our model, enriches this literature by studying the implications of ambiguity-aversion on dynamic liquidity management and investment.

The rest of the paper is organized as follows. Section 2 describes the basic setup of the model, the belief distortions, and the optimization problem. Section 3 solves the model. Section 4 analyzes the model and describes the paper's results. Section 5 concludes. All proofs are in the Appendix.

⁹Their results, albeit intriguing, hinge particularly on the use of Consensus Expected Utility (CEU) pioneered by Bewley (2002). This is an alternative preference framework based on multiple priors. The CEU choices are evaluated via a unanimity criterion under which one gamble is preferred to another if its expected value is higher under all priors within the set of priors.

2 The Model

2.1 The baseline model without model uncertainty

The baseline model builds on Bolton et al. (2011). Distinct from their model, we assume that firms' cash flows are exposed to both temporary and permanent shocks, as in Décamps et al. (2017) and Gryglewicz et al. (2018).

Consider a firm that generates a continuous stream of cash flows subject to temporary and permanent shocks. First, we assume that there is no agency frictions between the manager and the existing shareholders of the firm. That is, the manager makes decisions (i.e., chooses dividend, financing, and default policies) to maximize shareholder value. The absence of agency frictions implies that when shareholders are ambiguity-averse, the manager will exhibit the same preferences. We also assume that firms cannot access the bond market. Hence, shareholder value coincide with firm value. Therefore, “value” functions refer to the firm value function when the manager takes optimal actions to maximize expected discounted cash flows to the shareholders.

The manager is risk neutral and discounts future cash flows at the constant rate $r > 0$. The permanent and temporary shocks determining the firm's future cash flows are modeled via a two dimensional \mathbb{P} -Brownian processes $W_t = (W_t^P, W_t^T)$. The process (W_t) is defined on the standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where \mathbb{P} is the reference probability measure on the space. The standard filtration $\{F_t\}$ generated by $\sigma(W_s : s \leq t)$ satisfies the usual conditions.

Let firm size at time t be given by δ_t which evolves according to the controlled geometric brownian process:

$$d\delta_t = \delta_t [(\mu + i_t) dt + \sigma_\delta dW_t^P], \quad (1)$$

where μ is baseline growth rate net of depreciation, $\sigma_\delta > 0$ is the volatility of firm growth, and i_t is the firm's investment rate per unit of capital. In other words, the expected instant growth rate of firm's size depends on a constant drift μ and its (optimal) choice of investment. The shock process W_t^P represents permanent shocks to the firm's cash flows (justified later).

We assume that the firm's cumulative cash flows Y_t follows

$$dY_t = \delta_t \cdot [(\alpha - g(i_t)) dt + \sigma_Y dW_t^T], \quad (2)$$

where α and $\sigma_Y \in \mathbb{R}_+$. In particular, $g(i_t)$ represents the sum of the symmetric purchase/resale price of capital and the convex adjustment cost of capital. Henceforth, for tractability reasons we assume that $g(i) = i + \frac{\theta i^2}{2}$, where θ captures the adjustment cost of capital. We also assume the permanent shock W_t^P in (1) is orthogonal to the shock W_t^T that represents the temporary shock to firms' cash flows. Finally, we highlight the firm's cash flow in period t is proportional to its size at time t . Given this specification, the permanent shock W_t^P to the firm's size δ_t implies that an increase (decrease) in firm size raises (reduces) the expected value of the entire stream of future cash flows.

It is immediate that in the absence of short-term shocks (i.e., $\sigma_Y = 0$) firms' cash flows are always positive because firm size follows a geometric Brownian motion. Firms may have negative cash flows due to the short term shock W^T , which have to be covered either using cash reserves or raising external equity. Specifically, we allow firms to retain earnings and denote by M_t the firms' cash holding at time t . We assume cash reserves earn a constant interest rate $r - \lambda$. The constant $\lambda \in (0, r]$ denotes the carry cost of retained cash. The parameter λ captures agency costs associated with the free cash flow problem (Jensen, 1986).

We also allow a firm to increase its cash reserves by issuing new equity in the capital markets. Following Décamps et al. (2017), we assume that when firms issue an amount of new equity e_t at time t , the existing shareholders get $e_t/\rho_1 - \rho_0\delta_t$, where $\rho_1 > 1$ represents a proportional cost and $\rho_0\delta_t$ a fixed cost associated with the new equity issuance. For tractability, we model the fixed cost as being proportional to firm's size δ_t . The insights of our main results, however, do not rely on this assumption.

Finally, the firm's cash reserve process M_t follows

$$dM_t = (r - \lambda)M_t dt + dY_t + \frac{dE_t}{\rho_1} - d\Phi_t - dL_t, \quad (3)$$

where L_t is the cumulative dividend paid out to shareholders, E_t is the cumulative equity issuances, and Φ_t is the cumulative fixed cost of external financing. In other words, equation (3) states that the change in cash reserves is the sum of the net interest earned on a firm's cash holdings, its earnings, and the outside financing, less than the costs of the gross investment, and external financing, and the dividends paid out to existing shareholders. The gross cumulative financing E_t and the

cumulative cost of financing Φ_t are given by

$$\Phi_t = \sum_{n=1}^{\infty} \rho_0 \delta_{\tau_n} 1_{\tau_n \leq t} \quad \text{and} \quad E_t = \sum_{n=1}^{\infty} e_n 1_{\tau_n \leq t}. \quad (4)$$

Here $(\tau_n)_{n=1}^{\infty}$ represents an increasing sequence of the stopping times at which a firm issues new equity and $(e_n)_{n=1}^{\infty}$ a sequence of nonnegative random variables that represents the financing amounts.

A firm is liquidated if it is too costly to raise outside funds to cover its cash-reserve deficit. Let the (stopping) time of liquidation be $\tau_0 = \inf\{t \geq 0 | M_t = 0\}$. Upon liquidation, the shareholders receive $\omega V^A(\delta_{\tau_0})$ where $(1 - \omega)$ is a proportional deadweight cost due to liquidation and $V^A(\delta_{\tau_0})$ the firm value without financial frictions. As usual, $\tau_0 = \infty$ entails that liquidation does not occur.

2.2 Manager's belief distortions

In the baseline model, a manager is sure about the true model generating cash flows, in the sense he is certain the model is uniquely determined by (1) and (2).

The critical departure from the baseline model is, however, that the manager is *unsure* about the true model. Instead, he takes the baseline model ((1) and (2)) as the reference and considers a set of plausible models near the reference model. Moreover, he displays *aversion* to this model uncertainty (i.e., ambiguity-aversion).

To model managerial ambiguity-aversion, we use the recursive multiple prior model proposed by Chen and Epstein (2002).¹⁰ Formally, the manager obtains the set of plausible models by multiplying probabilities associated with (1) and (2) with a likelihood ratio. Following Chen and Epstein (2002) and Hansen et al. (2006), we represent the likelihood ratio by a martingale z^h (i.e., a density generator), which satisfies

$$dz_t^h = -z_t^h (h_t^P dW_t^P + h_t^T dW_t^T) \quad \text{and} \quad z_0^h = 1, \quad (5)$$

where the process (h_t^P, h_t^T) is progressively measurable with respect to the standard filtration. We adopt the common assumption that z_t^h is null when $\int_0^t |h_s|^2 ds$ is infinite. Assume the standard

¹⁰Roughly speaking, their model is the dynamic generalization of the max-min preference developed by Gilboa and Schmeidler (1989) when one allows ambiguity about the drift of the diffusion processes. For a model featuring ambiguity about the volatility of the diffusion processes see Epstein and Ji (2013, 2014).

Novikov condition is satisfied:

$$E \left[\exp \left(\frac{1}{2} \int_0^T |h_s|^2 ds \right) \right] < \infty; \quad 0 \leq T < \infty.$$

Imposing the initial condition $z_0^h = 1$, we express the solution of the stochastic differential equation (5) as the stochastic exponential

$$z_t^h = \exp \left\{ \frac{1}{2} \int_0^t |h_s|^2 ds - \int_0^t (h_s^P dW_s^P + h_s^T dW_s^T) \right\}, \quad 0 \leq t \leq T.$$

As specified, the process (z_t^h) is a \mathbb{P} -martingale.

The density h generates a probability measure \mathbb{Q}^h on (Ω, \mathcal{F}) that is equivalent to the reference measure \mathbb{P} such that

$$\mathbb{Q}^h(A) = \mathbb{E} \left[1_A z_t^h \right] \equiv \int 1_A z_t^h d\mathbb{P}, \quad \text{for all } A \in \mathcal{F} \quad (6)$$

In other words, any probability measure \mathbb{Q}^h and the reference \mathbb{P} should agree on the set of zero-probability events. Because $E^{\mathbb{Q}^h}[z_T^h] = z_0^h = 1$ and $z_t^h > 0$ for all t , the Radon-Nykodym theorem ensures a choice of (h_t) generates a (conditional) probability Q^h on (Ω, \mathcal{F}) ,

$$\left. \frac{d\mathbb{Q}^h}{d\mathbb{P}} \right|_{\mathcal{F}_t} = z_t^h. \quad (7)$$

Note that $h = 0$ represents the case where the manager faces no model uncertainty about the model, that is, he is absolutely sure about the reference \mathbb{P} .

Finally, to construct the set of priors that the manager thinks plausible, it is necessary to specify the set of density generators. Following Chen and Epstein (2001), we assume that the set of priors satisfy IID-Ambiguity.¹¹ Define the set H such that

$$H = \left\{ (h_t) : \sup \left\{ \left| \frac{h_t^P}{\kappa_P} \right|^\gamma + \left| \frac{h_t^T}{\kappa_T} \right|^\gamma : 0 \leq t \leq T \right\} \leq 1 \right\}. \quad (8)$$

For a given H , we construct the set of priors P^H such that

$$P^H = \left\{ \mathbb{Q}^h : h \in H \text{ and } \mathbb{Q}^h \text{ is defined by (7)} \right\}. \quad (9)$$

¹¹See the details in Section 3.4 of Chen and Epstein (2001).

By Girsanov's theorem (Karatzas and Shreve, 1991), we generate a standard two dimensional Q^h -Brownian motion, $\hat{W} = (\hat{W}_t^P, \hat{W}_t^T)$:

$$\begin{bmatrix} \hat{W}_t^P \\ \hat{W}_t^T \end{bmatrix} = \begin{bmatrix} W_t^P \\ W_t^T \end{bmatrix} - \begin{bmatrix} \int_0^t h_s^P ds \\ \int_0^t h_s^T ds \end{bmatrix}. \quad (10)$$

In light of (10), we can write models (1) and (2) as:

$$d\delta_t = \delta_t \left[(\mu + i_t + \sigma_\delta h_t^P) dt + \sigma_\delta \hat{W}_t^P \right], \quad (11)$$

and

$$dY_t = \delta_t \cdot [(\alpha - g(i_t) + \sigma_Y h_t^T) dt + \sigma_Y d\hat{W}_t^T]. \quad (12)$$

That is, a manager concerned about model uncertainty considers multiple models for the dynamics of firm size and cumulative cash flows (11) and (12). A given density generator $(h_t^P, h_t^T) \in H$ indexes a particular model among such candidates.

2.3 Discussions on the set H

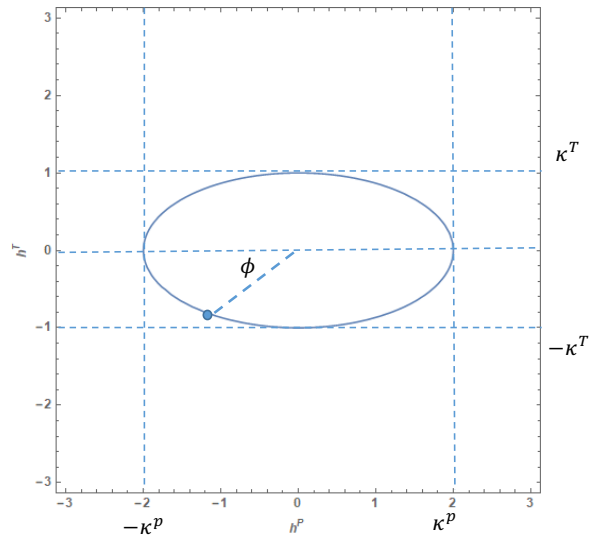


Figure 2.1: ℓ^2 -norm ($\gamma = 2$)

From the definition of the set H in (8), the parameters κ_P and κ_T capture the amount am-

biguity born by the manager regarding the distribution of the permanent and temporary shocks, respectively. As κ_P (κ_T) increases, the manager considers a larger set of plausible models driving the the permanent (temporary) shock process. In other words, increases in κ_t^T and κ_t^P make the manager less confident about the reference model.

For tractability, we restrict ourselves to the ℓ^2 -norm by setting $\gamma = 2$. Thus, from (8), the boundary of set H given by

$$\partial H = \left\{ (h_t^P, h_t^T) : \left| \frac{h_t^P}{\kappa_P} \right|^2 + \left| \frac{h_t^T}{\kappa_T} \right|^2 = 1 \right\} \quad (13)$$

is an ellipsoid, as illustrated in Figure 2.1. Therefore, the set H consists of all points inside the boundary ∂H (i.e., inside the ellipsoid). We choose the ℓ^2 -norm to preserve a smooth worst-case beliefs allocation between the temporary and permanent components. Moreover, all points on the boundary ∂H can be uniquely identified by the angle ϕ using polar coordinates

$$\partial H := \{ (h^P, h^T) = (-\kappa_P \cos \phi, -\kappa_T \sin \phi) : 0 \leq \phi < 2\pi \},$$

which simplify our analysis.

2.4 Ambiguity-averse manager's optimization problem

The two state variables for the manager's optimization problem are firm size δ_t and cash reserves M_t . The ambiguity-averse manager solves

$$[\mathcal{P}_0] \quad V(\delta, M) = \max_{\tau_n, e_n, L, i} \min_{\mathbb{Q}^h \in P(H)} \mathbb{E}^{\mathbb{Q}^h} \left[\int_0^{\tau_0} e^{-rt} (dL_t - dE_t) + e^{-r\tau_0} \omega (V^A(\delta_{\tau_0}) + M_{\tau_0}) \right] \quad (14)$$

subject to (3), (4), (11), and (12).

The first term in (14) represents the net present value of payments to existing shareholders until liquidation time τ_0 , net of the claims of new equity-holders on future cash flows. The second term captures the firm's discounted liquidation value.

The manager's ambiguity-aversion is captured by the fact that the expectation $\left(\min_{\mathbb{Q}^h \in P(H)} E^{\mathbb{Q}^h} [\cdot] \right)$ is taken with respect to their worst-case beliefs. Again, note that when a manager faces no model uncertainty, the set $P(H)$ becomes a singleton.

3 Model Solution

In this section, we first solve the model without managerial ambiguity-aversion or financial frictions. Next, we characterize the model with ambiguity aversion alone. Finally, we solve the full model with financial frictions and ambiguity-aversion (i.e., the problem \mathcal{P}_0).

3.1 The baseline model without ambiguity-aversion and financial frictions

We first consider the model of a manager facing neither financial frictions nor model uncertainty.¹² Precisely, the baseline model is obtained by setting $\rho_0 = \rho_1 = 0$ (no costs of raising external financing) and $\kappa_T = \kappa_P = 0$ (no model uncertainty) in the complete problem \mathcal{P}_0 . This benchmark helps us develop a clear understanding of the role played by financial frictions and ambiguity-aversion.

Proposition 1. *The value function when the manager can costlessly raise equity and faces no model uncertainty is given by $V^{FB}(\delta) = q^{FB}\delta$, where the marginal value of capital q^{FB} and the investment rate i^{FB} solve*

$$q^{FB} = \frac{\alpha - g(i^{FB})}{r - \mu - i^{FB}}, \quad \text{and} \quad i^{FB} = \frac{q^{FB} - 1}{\theta}.$$

where $g(i) = i + \theta i^2/2$.

All proofs are in the Appendix.

Proposition 1 recovers the results in Hayashi (1982). In the absence of financial frictions and model uncertainty, firm value is linear in the firm's size δ , rendering the average and marginal value of capital equal to q^{FB} . Therefore, the optimal investment is proportional to the marginal benefit of increasing the firm size by one unit, scaled by the adjustment costs.

3.2 The model with ambiguity-aversion but without financial frictions

In this section we characterize the solution to the model when a manager is not constrained by financing frictions but is ambiguity-averse. To do so, we first assume $\rho_0 = \rho_1 = 0$ (i.e., no costs of raising external), and in order to capture model uncertainty, we set $\kappa_T > 0$ and/or $\kappa^P > 0$ in (8).

¹²Alternatively, we can think of this baseline as the manager being ambiguity-neutral to any given model uncertainty.

Recall that equation (8) characterizes the set of density generators that the manager minimizes over. In particular, since we focus on the case $\gamma = 2$, we can parametrize the perimeter of set H_t using polar coordinates instead of Cartesian ones:

$$\partial H_t := \{(h_t^P, h_t^T) = (-\kappa_P \cos \phi_t, -\kappa_T \sin \phi_t) : 0 \leq \phi_t \leq 2\pi\},$$

Figure 2.1-(a) illustrates how the angle ϕ_t indexes an element in the set ∂H_t . Effectively, we can think of ϕ_t as the optimal allocation of “pessimism” between temporary and permanent shocks: the higher ϕ_t , the manager is relatively more pessimistic about the temporary than the permanent shocks.¹³

In the Appendix, we show that without loss of generality, minimizing on the boundary ∂H_t is equivalent to minimizing over the closure of the set H_t . We enhance the tractability of the problem thanks to this simplification. Denoting the value function for this model by $V^A(\delta)$, the HJB reduces to

$$rV^A(\delta) = \max_i \min_{\phi \in [0, 2\pi]} \left\{ \delta(\alpha - g(i) - \sigma_Y \kappa_T \sin \phi) + V_\delta^A(\delta) \delta(\mu + i - \kappa_P \sigma_\delta \cos \phi) + \frac{1}{2} V_{\delta\delta}^A(\delta) \sigma_\delta^2 \delta^2 \right\}, \quad (15)$$

where the $\min(\cdot)$ operator characterizes the manager’s ambiguity-aversion toward model uncertainty as measured by ϕ_t (for fixed κ_T and κ_P).

Proposition 2. *When managers raise equity costlessly but face ambiguity-aversion, the value function is given by $V^A(\delta) = q^A \delta$. The marginal value of capital q^A , the investment rate i^A , and the optimal allocation of worst-case beliefs ϕ^A jointly solve*

$$r q^A = (\alpha - g(i^A)) - \delta \cdot \sigma_Y \cdot \kappa_T \cdot \sin \phi^A + q^A \cdot (\mu + i^A - \kappa_P \cdot \sigma_\delta \cdot \cos \phi^A), \quad (16)$$

$$i^A = \frac{q^A - 1}{\theta}, \quad \text{and} \quad \phi^A = \arctan \left(\frac{1}{q^A} \cdot \frac{\sigma_Y \kappa_T}{\sigma_\delta \kappa_P} \right). \quad (17)$$

where $\arctan \equiv \tan^{-1}$. The worst-case beliefs are given by $(h_A^P, h_A^T) = (-\kappa_P \cos \phi^A, -\kappa_T \sin \phi^A)$. Moreover, the investment rate and firm value under ambiguity are lower than the first best bench-

¹³Recall that ϕ_t measures the counter-clockwise angle from the $(-\kappa^P, 0)$ -axis.

mark. That is,

$$q^A < q^{FB} \quad \text{and} \quad i^A < i^{FB}.$$

Relative to the baseline model, the ambiguity-averse manager is pessimistic about the firm's future cash flows. Naturally, this leads to lower values of capital and optimal investment rates than in the baseline model. In other words, even in the absence of financial frictions, the model augmented with ambiguity-aversion induces under-investment.

Remark 1. *Proposition 2 is a special case of the full model characterized in Proposition 3. That is, $1/q^A$ is interpretable as the marginal rate of substitution between cash and capital. In the absence of costly external finance, there is no distinction between internal and external cash. Therefore, the marginal value of cash should be equal to one, whereas that of capital is the shadow value of capital q^A .*

Roughly speaking, equation (17) shows the relative importance of model uncertainty imposed on permanent versus temporary shocks: higher model uncertainty with respect to the temporary component κ_T implies a higher ϕ^A , and managers become more pessimistic with respect to the temporary shocks.¹⁴ Moreover, the higher the marginal value of capital q^A , the more the investor becomes concerned about permanent shocks.

Asset Pricing and Ambiguity Premia

Finally, we compute the equity premium implied in the ambiguity benchmark. By a similar analysis as in Anderson, Hansen and Sargent (2003), we can show that the shareholders ambiguity-aversion generates a market price of model uncertainty. This market price of model uncertainty can be decomposed into the permanent component $|h_t^P|$ and the temporary component $|h_t^T|$ of the worst-case density generator given in (17). Both of these components contribute to the equity premium.

We calculate the excess return an ambiguity neutral agent would expect to get from holding

¹⁴Formally, we can compute the derivative of ϕ^A with respect to κ_T and κ_P . It is straightforward to show that ϕ^A is increasing in κ_T . In contrast, the derivative of ϕ^A with respect to κ_P is analytically ambiguous. However, our numerical results show that its sign is negative for a wide range of parameter values.

the firm's equity denoted by ep^A :

$$\underbrace{E_t^P \left[\frac{dY_t + dV^A(\delta_t)}{V^A(\delta_t)} - r \right]}_{ep^A} = E_t^P \left[\frac{\delta_t ((\alpha - g(i^A)) dt + \sigma_Y dW_t^T) + q^A (\delta_t (\mu + i^A) dt + \sigma_\delta dW_t^P)}{q^A \delta_t} - r \right]$$

$$= \underbrace{\sigma_\delta \kappa_P \cos \phi^A}_{ep_P^A} + \underbrace{\frac{\sigma_Y \kappa_T \sin \phi^A}{q^A}}_{ep_T^A}$$

where ep_P^A is the premium due to ambiguity with respect to the permanent shocks and ep_T^A is the premium due to ambiguity with respect to the temporary shocks. The equity premia in this case is constant, and is proportional to the degree of ambiguity perceived with respect to each of the shocks. In the next section we show that financial frictions induce endogenous time variation in the ambiguity premia, thereby linking equity premia and financial distress.

3.3 The full model with ambiguity-aversion and financial frictions

In this section we use heuristic arguments to solve the full model with managerial ambiguity-aversion and financial frictions (14).¹⁵

Due to fixed costs, managers access equity markets only if they have depleted their cash reserves. Depending on financing costs, the manager either issues new equity or liquidates the firm. When it comes to payout decisions, the marginal benefit of an additional dollar of cash reserves is decreasing. By contrast, the marginal cost of retaining cash is increasing due to the carry cost. Thus, we conjecture that there exists a size-dependent cash-holding threshold $M^*(\delta)$ at which dividends are optimally paid out.

First, consider the region $(0, M^*(\delta))$ over which it is optimal to retain earnings ($dE_t = dL_t = 0$). Hence the existing shareholders value function satisfies:

$$rV(\delta, M) = \max_i \min_{\phi \in [0, 2\pi]} \left\{ \begin{array}{l} V_\delta(\delta, M) \delta (\mu + i - \sigma_\delta \kappa_P \cos \phi) \\ + V_M(\delta, M) (\delta (\alpha - g(i) - \sigma_Y \kappa_T \sin \phi) + (r - \lambda) M) \\ + \frac{1}{2} V_{\delta\delta}(\delta, M) \sigma_\delta^2 \delta^2 + \frac{1}{2} V_{MM}(\delta, M) \sigma_Y^2 \delta^2 \end{array} \right\}. \quad (18)$$

The left hand side corresponds to the required return on investment for the shareholders. The

¹⁵We formalize these arguments in Appendix A.

right hand side corresponds to the shareholders' expected gain under the worst-case belief in the no-payout region. The first two terms capture the changes in equity value induced by the change in firm size and cash reserves. The last two terms capture the effect of volatilities on firm size and cash reserves on the value function.

Proposition 3. *The optimal investment i and allocation of worst-case beliefs ϕ satisfy*

$$\delta \cdot V_M(\delta, M) \cdot \underbrace{(1 + \theta \cdot i(\delta, M))}_{\equiv g'(i(\delta, M))} = V_\delta(\delta, M) \quad \text{and} \quad \phi(\delta, M) = \arctan \left(\frac{V_M(\delta, M)}{V_\delta(\delta, M)} \cdot \frac{\sigma_Y \kappa_T}{\sigma_\delta \kappa_P} \right). \quad (19)$$

The first equation shows the marginal cost of investment depends on the marginal value of cash $V_M(\delta, M)$ due to financial frictions. Because external finance is expensive, financial frictions make investment costlier than in the frictionless baseline.

The implication of the second condition is critical: when the manager's marginal rate of substitution of cash for capital ($MRS_{\delta, M} = V_M/V_\delta = d\delta/dM$) is high, he is more pessimistic toward temporary cash flows shock. That is, when the marginal value of liquidity is high, the manager is more concerned about the possibility of inefficient liquidation due to operating losses (i.e., negative temporary shock realizations). Conversely, when he holds sufficient cash, such that $MRS_{\delta, M}$ is relatively low, the manager is more concerned about the long-term prospects of the firm (i.e., about permanent shocks).

The value function (18) has to be solved subject to the following boundary conditions. At the payout boundary $M^*(\delta)$, the marginal value of cash is equal to 1:

$$V_M(\delta, M^*(\delta)) = 1. \quad (20)$$

In addition, the function V requires "smoothness": twice differentiability. This implies the super contact condition:

$$V_{MM}(\delta, M^*(\delta)) = 0. \quad (21)$$

Together, these two conditions determine the location of the payout boundary.

Second, let the optimal equity issuance boundary be $\hat{M}(\delta)$. At this boundary, the marginal benefit of an additional unit of cash is equal to the marginal cost of raising external finance. Hence,

the optimal boundary $\hat{M}(\delta)$ must satisfy the first order condition:

$$V_M(\delta, \hat{M}(\delta)) = \rho_1 \quad (22)$$

Finally, when the cost of raising outside equity is too high, a manager may find it optimal to liquidate the firm instead. We assume that upon default, the new shareholders are also ambiguity-averse, but do not face financial frictions:

$$V(\delta, 0) = \omega \delta \frac{(\alpha - g(i^A))}{r - \mu - i^A} \equiv \omega \delta q^A, \quad (23)$$

where the proportional deadweight loss $1 - \omega$ captures inefficient liquidation.

Even though there are two state variables in problem (18), we have made assumptions to ensure the problem is homogeneous of degree one in δ and M . Consequently, the value function can be re-expressed using the single state variable $m = \frac{M}{\delta}$ that represents the scaled cash holdings per unit of δ :

$$V(\delta, M) = \delta V\left(1, \frac{M}{\delta}\right) = \delta F(m), \quad (24)$$

where $F(m)$ represents the equivalent scaled value function. Using this scaled value function, we re-write the aforementioned boundary conditions as a standard one-dimensional free boundary problem, where the scaled cash holdings evolve between the liquidation/refinancing boundary at $m = 0$ and the payout boundary m^* .

We now substitute (24) into equations (18) to obtain the scaled value function:

$$rF(m) = \max_i \min_{\phi \in [0, 2\pi]} \left\{ \begin{array}{l} (F(m) - F'(m)m)(\mu + i - \sigma_\delta \kappa_P \cos \phi) \\ + F'(m)(\alpha - g(i) - \sigma_Y \kappa_T \sin \phi + (r - \lambda)m) \\ + \frac{1}{2} F''(m)(\sigma_Y^2 + m^2 \sigma_\delta^2) \end{array} \right\}. \quad (25)$$

The FOCs (19) become

$$i(m) = \frac{F(m) - F'(m)(m+1)}{\theta F'(m)}, \quad \text{and} \quad \phi(m) = \arctan\left(\frac{F'(m)}{F(m) - F'(m)m} \cdot \frac{\sigma_Y \kappa_T}{\sigma_\delta \kappa_P}\right) \quad (26)$$

and the worst-case beliefs are given by

$$h^P(m) = -\kappa_P \cos \phi(m), \quad \text{and} \quad h^T(m) = -\kappa_T \sin \phi(m). \quad (27)$$

Corresponding to (20) and (21), the payout boundary m^* satisfies:

$$F'(m^*) = 1, \quad \text{and} \quad F''(m^*) = 0. \quad (28)$$

When the firm runs out of cash, the manager chooses between liquidation and equity issuance, therefore the scaled value function satisfies:

$$F(0) = \max \left\{ \max_m \{F(m) - \rho_1(m + \rho_0)\}, \omega q^A \right\}. \quad (29)$$

When issuing new equity is optimal, the optimal amount of (scaled) cash satisfies the first order condition:

$$F'(\hat{m}) = \rho_1. \quad (30)$$

Finally, in the payout region $m > m^*$, the firm pays out excess cash above the optimal payout boundary m^* :

$$F(m) = F(m^*) + (m - m^*) \quad \forall m > m^*. \quad (31)$$

Asset Pricing and Ambiguity Premia

We proceed in a similar way as in the previous section to calculate the ambiguity premia implied by the model. The excess return an ambiguity neutral agent would expect to get from holding the firm's equity denoted by $ep(m_t)$ is given by

$$\begin{aligned} \underbrace{E_t^P \left[\frac{dV(\delta_t, M_t)}{V(\delta_t, M_t)} - r \right]}_{ep(m_t)} &= E_t^P \left[\frac{dF(m_t)}{F(m_t)} - r \right] \\ &= \underbrace{\frac{\sigma_\delta \kappa_P \cos \phi(m_t) (F(m_t) - F'(m_t)m_t)}{F(m_t)}}_{ep^P(m_t)} + \underbrace{\frac{\sigma_Y \kappa_T \sin \phi(m_t) F'(m_t)}{F(m_t)}}_{ep^T(m_t)}, \end{aligned} \quad (32)$$

where $ep^P(m_t)$ is the premium due to ambiguity with respect to the permanent shocks and $ep^T(m_t)$ is the premium due to ambiguity with respect to the temporary shocks. The equity premium associated with each source of uncertainty is time varying, and depends on the state variable m .

4 Model Analysis

Following Décamps et al. (2017), we set $\mu = 0.01$, $r = 0.06$, $\alpha = 0.17$, $\lambda = 0.01$, $\sigma_\delta = 0.25$, $\sigma_Y = 0.12$, $\omega = 0.55$, $\rho_0 = 0.002$, and $\rho_1 = 1.06$. We calibrate the adjustment cost parameter $\theta = 150$, to obtain 2% net investment rate in the first best benchmark (i.e., $i^{FB} = 0.02$). We vary the ambiguity-aversion parameters $\kappa_T \in [0, 0.5]$ and $\kappa_P \in [0, 0.2]$. For simplicity, in the following sections, we assume that firms are not allowed to issue new equity. The firm's equity issuance will be extensively discussed in section 4.3.

4.1 Time varying optimal worst-case beliefs and extrapolation bias

We characterize the endogenous time variation in the manager's optimal worst-case beliefs of permanent and temporary shocks. We show that time-varying ambiguity aversion induces the manager to behave *as if* he displayed extrapolation bias. When the manager experiences a series of positive (negative) cash flows, he becomes more optimistic (pessimistic) about the firm's future cash flows. That is, the manager forms beliefs about the future by extrapolating from the recent past.

Proposition 3 previews this intuition. Depending on the magnitude of the marginal value of liquidity, the manager is more concerned by either the temporary or the permanent cash flow model uncertainty. For example, suppose that a series of negative temporary cash flow shocks depletes the firm's cash reserves. As a result, the manager's marginal value of liquidity increases, and his worst-case beliefs feature more pessimism toward the distribution of temporary shocks than towards the permanent shocks.

In order to feature ambiguity with respect to both shocks, we set $\kappa_P = 0.05$ and $\kappa_T = 0.5$. Panel A and D of Figure (4.1) depict the manager's optimal worst-case beliefs about the permanent ($h^P(m)$) and the temporary shocks ($h^T(m)$). We make two important observations from these figures:

1. Adding costly external equity issuance to the model with managerial ambiguity-aversion

makes the manager unconditionally more concerned about model uncertainty with respect to temporary shocks.

2. Throughout the life-time of the firm, the manager's optimal ambiguity-aversion toward both shocks is time varying. In particular, as the firm gets close to liquidation (i.e., when the marginal value of liquidity is high), the manager is particularly concerned about the distribution of the short-term temporary shocks, and exhibits relatively little concern about the distribution of long-term permanent shocks.

The economic intuition is the following. First, it is costly for the firm to run out of cash, since raising equity is costly and liquidation is inefficient. Facing low cash reserves, the ambiguity-averse manager becomes more concerned about model uncertainty with respect to temporary shocks, because they may trigger liquidation. Importantly, in the baseline model with ambiguity-aversion but without financial frictions, the manager's beliefs are time invariant, given that the marginal value of liquidity is always equal to 1. As a result, our model implies that $h^P(m) > h_A^P$ and $h^T(m) < h_A^T$.

Second, as the firm's financial situation improves (i.e., the cash to capital ratio increases), the manager is less worried about the threat of liquidation. Hence, his worst-case scenario features more pessimism about the long-term prospects of the firm (i.e., about permanent shocks). Because negative temporary cash flow shocks can not trigger immediate liquidation when the firm has sufficiently large cash reserves, he displays less concern about these shocks. Therefore, $h^P(m)$ is increasing in m , while $h^T(m)$ is decreasing in m .

Finally, we explore the sensitivity of the manager's optimal worst-case allocation to the permanent and temporary shocks. Define the sensitivity of $h^P(m)$ to permanent shocks as the change in $dh^P(m)$ induced by a unit change in the permanent shocks dW^P . In other words, the sensitivity measures how much the manager adjusts his optimal worst-case beliefs allocation toward model uncertainty of the permanent shocks when the firm experiences an impulse to its permanent shocks. We define the sensitivity of $h^T(m)$ in a similar manner.

We turn to investigate the responses of the manager's optimal allocation of worst-case beliefs with respect to a unit impulse of temporary and permanent shocks. Using Ito's formula, we can

compute the dynamics of $h^P(m)$ and $h^T(m)$ to obtain:

$$\begin{aligned} \langle d(h^P(m_t)), dW_t^P \rangle &= -\frac{dh^P(m_t)}{dm} m \sigma_\delta dt & \text{and} & \quad \langle d(h^P(m_t)), dB_t^T \rangle = \frac{dh^P(m_t)}{dm} \sigma_Y dt, \\ \langle d(h^T(m_t)), dW_t^P \rangle &= -\frac{dh^T(m_t)}{dm} m \sigma_\delta dt & \text{and} & \quad \langle d(h^T(m_t)), dB_t^T \rangle = \frac{dh^T(m_t)}{dm} \sigma_Y dt. \end{aligned}$$

Panels B, C, E, and F in Figure (4.1) depict these quantities. The critical insight from this analysis is that the sensitivity of h^P to permanent (temporary) shocks is positive (negative), while the sensitivity of h^T to permanent (temporary) shocks is negative (positive). Therefore, we claim that our model generates *endogenous extrapolation bias*, that overweights recent realizations when forming beliefs about future shocks. For example, if the manager observes a stream of high cash flows, he becomes more optimistic about the firm's future cash flows (but less optimistic about its long-term growth). On the other hand, if he experiences high growth rates, then the manager becomes more optimistic about the firm's growth prospects (but less optimistic about future short-term cash flows). This result is interpretable as an endogenous link between ambiguity-aversion and extrapolation bias. Hence, our model provides *a microeconomic foundation* for managerial extrapolation bias. We believe that this insight is novel and unexpected, since ambiguity-aversion and extrapolation bias are typically used to rationalize very different types of behavior.¹⁶ We clarify, however, that we do not claim a general link between ambiguity aversion and extrapolation bias. The connection is specific to our modeling assumptions.¹⁷

4.2 Implications for Cash Policy

In this section we explore the implications of ambiguity-aversion for the cash and dividend policy of the firm. We show that ambiguity with respect to permanent shocks has the opposite effect on the firm's dividend policy than ambiguity with respect to temporary shocks. By contrast, volatility (or riskiness) of permanent shocks has the same effect on the firm's dividend policy as volatility of temporary shocks. Thus our model provides implications that are unique to the ambiguity aversion

¹⁶Ambiguity-aversion is concerned with the agent's desire to protect himself against model mis-specification, while extrapolation bias captures the agent's bias of over-emphasizing recent events. Hence, there is no a priori reason to expect a relation between them.

¹⁷We *conjecture*, however, that this link is present in contexts featuring: i) dynamic ambiguity-aversion with time varying worst-case beliefs, and ii) higher marginal losses from a negative shock conditional on a previous sequence of bad shocks.

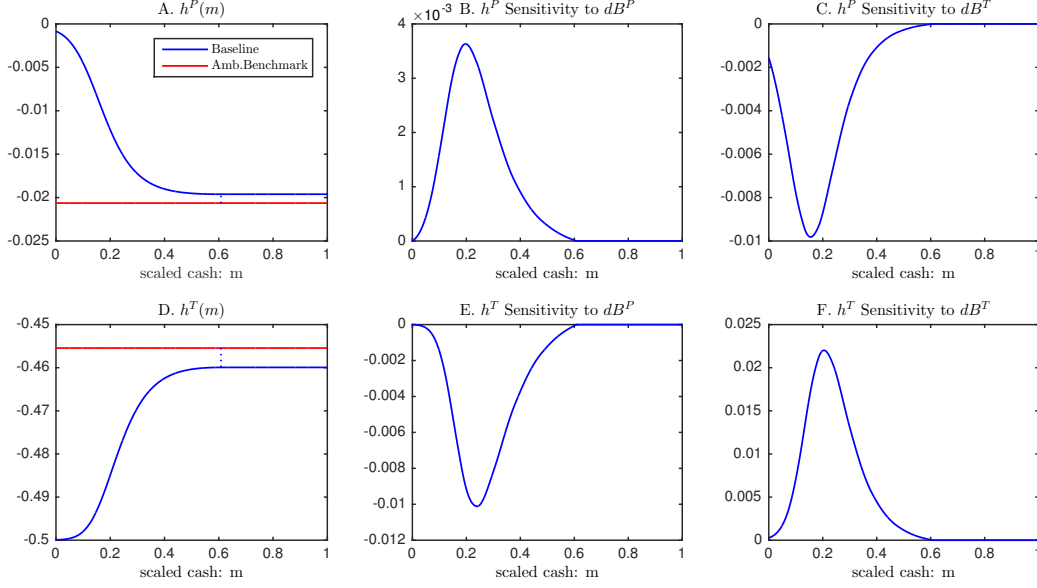


Figure 4.1: Belief Distortions $h^P(m)$ and $h^T(m)$, and sensitivities to permanent and temporary shocks. Parameter values are $\mu = 0.01, r = 0.06, \alpha = 0.17, \lambda = 0.01, \sigma_\delta = 0.25, \sigma_Y = 0.12, \omega = 0.55, \theta = 150, \kappa_P = 0.05, \kappa_T = 0.5$.

model, that a model exclusively featuring risk cannot capture.

Figure (4.2) depicts comparative statics of the firm value $F(m)$ and the payout boundary \bar{m} with respect to an increase in the ambiguity of permanent shocks κ_P , and the volatility of permanent shocks σ_δ . Panel A (C) confirms our intuition that higher ambiguity (volatility) reduces firm value. Higher κ_P distorts downwards the firm's growth rate, thus making it less valuable. Similarly, higher σ_δ makes it more likely for the firm to run out of cash, thereby triggering inefficient liquidation, and reducing firm value. However, Panel B (D) shows that higher ambiguity (volatility) with respect to permanent shocks reduces (increases) the firm's payout boundary \bar{m} . Increasing κ_P makes the manager pessimistic with respect to the growth rate of δ , and thus he thinks it is less likely to run out of cash per unit of δ . Intuitively, the manager doesn't see the point of hoarding cash to protect an asset base that is unlikely to grow large. Therefore, higher κ_P reduces the precautionary motive for accumulating cash.¹⁸ In contrast, increasing σ_δ makes the dynamics of m more volatile, making it more likely for the firm to face inefficient liquidation. In anticipation of this event, the manager optimally increases his precautionary cash reserves.

¹⁸Breuer et al. (2016) obtain a similar result in a static framework. Our contrasting results between temporary and permanent ambiguity underscore the importance of a dynamic model that allows for a distinction between shocks that are permanent in nature and those that are only temporary.

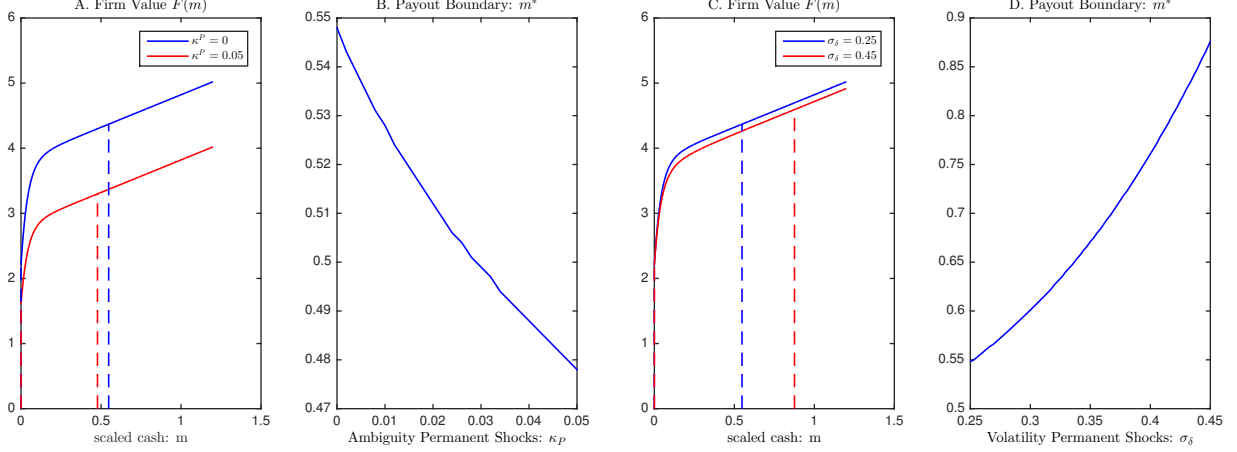


Figure 4.2: Comparative Statics for κ_P and σ_δ . Panel A depicts comparative statics of firm value with respect to κ_P . Panel B computes comparative statics for the payout boundary with respect to κ_P . Panel C depicts comparative statics of firm value with respect to σ_δ . Panel D computes comparative statics for the payout boundary with respect to σ_δ . Parameter values are $\mu = 0.01, r = 0.06, \alpha = 0.17, \lambda = 0.01, \sigma_\delta = 0.25, \sigma_Y = 0.12, \omega = 0.55, \theta = 150, \kappa_T = 0$.

Figure (4.3) depicts comparative statics of the firm value $F(m)$ and the payout boundary \bar{m} with respect to an increase in the ambiguity of temporary shocks κ_T and the volatility of temporary shocks σ_Y . Panel A (C) confirms the intuition that higher ambiguity (volatility) reduces firm value. Higher κ_T distorts downward the firm's expected cash flows, thus making it less valuable. Similarly, higher σ_Y makes it more likely for the firm to run out of cash, triggering inefficient liquidation. Moreover, panels B and D show that \bar{m} is increasing in both κ_T and σ_Y . Higher κ_T makes the manager pessimistic about future cash flows and the possibility his assets will not generate enough internal liquidity to withstand adverse cash flow realizations. Thus, he prefers to delay his dividend payout in order to protect his existing assets and reduce the probability of inefficient liquidation. Similarly, higher σ_Y increases the volatility of m , thereby rendering optimal for the manager to increase his precautionary cash reserves.

4.3 Implications for Equity Issuance

In this section we assume shareholders can raise equity in the capital markets in order to replenish their cash reserves. We study the implications of ambiguity-aversion on the equity issuance decision. Recall the firm faces both a fixed cost ρ_0 and a proportional cost ρ_1 when raising fresh equity. Once the firm runs out of cash, shareholders will increase their cash reserves by a lump-sum \hat{m} , which

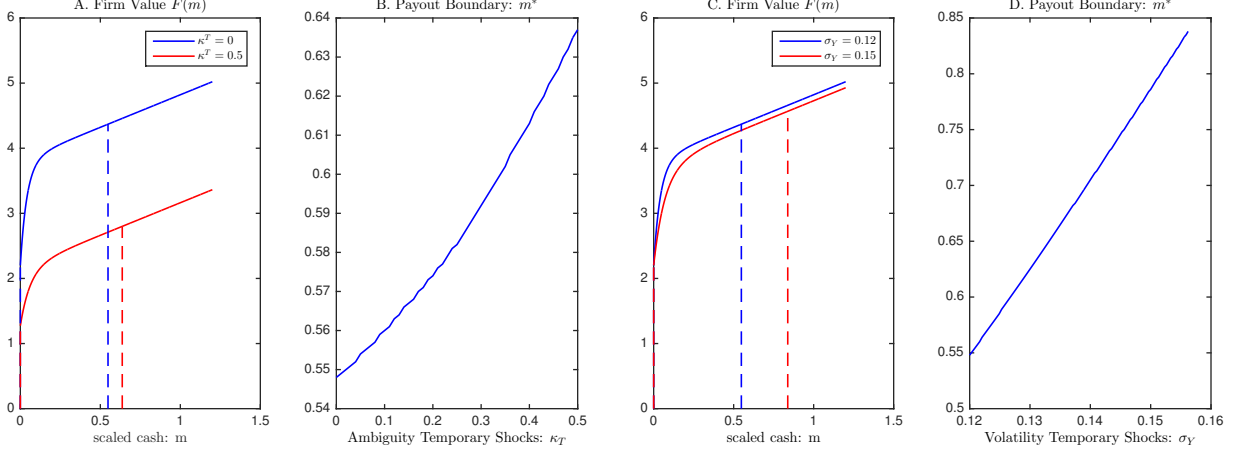


Figure 4.3: Comparative Statics for κ_T and σ_Y . Panel A depicts comparative statics of firm value with respect to κ_T . Panel B computes comparative statics for the payout boundary with respect to κ_T . Panel C depicts comparative statics of firm value with respect to σ_Y . Panel D computes comparative statics for the payout boundary with respect to σ_Y . Parameter values are $\mu = 0.01, r = 0.06, \alpha = 0.17, \lambda = 0.01, \sigma_\delta = 0.25, \sigma_Y = 0.12, \omega = 0.55, \theta = 150, \kappa_P = 0$.

we refer to as the refinancing target.

Panel A of Figure 4.4 depicts comparative statics of the firm’s value function and their respective payout boundaries and refinancing targets with respect to the degree of ambiguity towards permanent shocks κ_P . Consistent with the results in the previous section, increasing κ_P reduces firm value and encourages the firm to pay dividends earlier. Moreover, the firm raises smaller amounts of cash, since pessimism about future growth reduces the precautionary incentive to hoard cash to protect future assets. Panel B confirms that the relationship between κ_P and m^* and \hat{m} is monotonically decreasing.

Similarly, Panel C depicts comparative statics with respect to the degree of ambiguity towards temporary shocks κ_T . Increasing κ_T reduces firm value and encourages the firm to delay dividend payments. These results are consistent with our findings when refinancing is not allowed. Furthermore, equity issues are larger, compared to the ambiguity neutral case. Pessimism about cash flow shocks, means shareholders are concerned about the possibility of negative cash flow realizations that force the firm to pay for costly equity issues. In anticipation of that shareholders prefer to raise larger amounts of cash.

In conclusion, this section confirms our findings that ambiguity-aversion with respect to permanent and temporary shocks have opposite implications for firm’s cash policy. In particular,

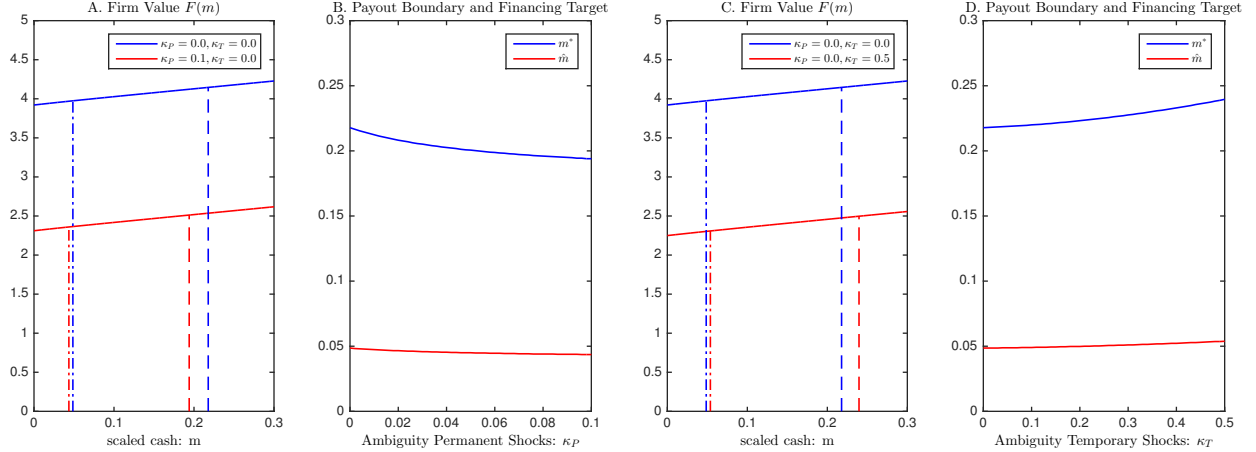


Figure 4.4: Comparative Statics for κ_P and κ_T . Panel A depicts comparative statics of firm value with respect to κ_P . Panel B computes comparative statics for the payout boundary m^* and the refinancing target \hat{m} with respect to κ_P . Panel C depicts comparative statics of firm value with respect to κ_T . Panel D computes comparative statics for the payout boundary m^* and the refinancing target \hat{m} with respect to κ_T . Parameter values are $\mu = 0.01, r = 0.06, \alpha = 0.17, \lambda = 0.01, \sigma_\delta = 0.25, \sigma_Y = 0.12, \omega = 0.55, \theta = 150, \kappa_T = 0, \rho_0 = 0.002, \rho_1 = 1.06$.

increasing ambiguity with respect to permanent (temporary) shocks reduces (increase) the firm's payout boundary m^* and the firm's refinancing target \hat{m} . In contrast, increasing the volatility of either the permanent or temporary shocks increases m^* and \hat{m} . Thus, empirical studies focusing on the effect of uncertainty on the firm's cash policy, need to distinguish changes in volatility versus changes in uncertainty (ambiguity).

4.4 Implications for Investment

In this section we study the joint effect of ambiguity-aversion and financial frictions in the investment policy of the firm. Figure (4.5) depicts the firm's investment policy for our three benchmark models:

1. The first best investment policy i^{FB} when managers are ambiguity neutral and do not face financial frictions.
2. The ambiguity benchmark investment policy i^A when managers are ambiguity-averse, but do not face financial frictions.
3. The investment policy $i(m)$ when managers are ambiguity-averse and face financial frictions.

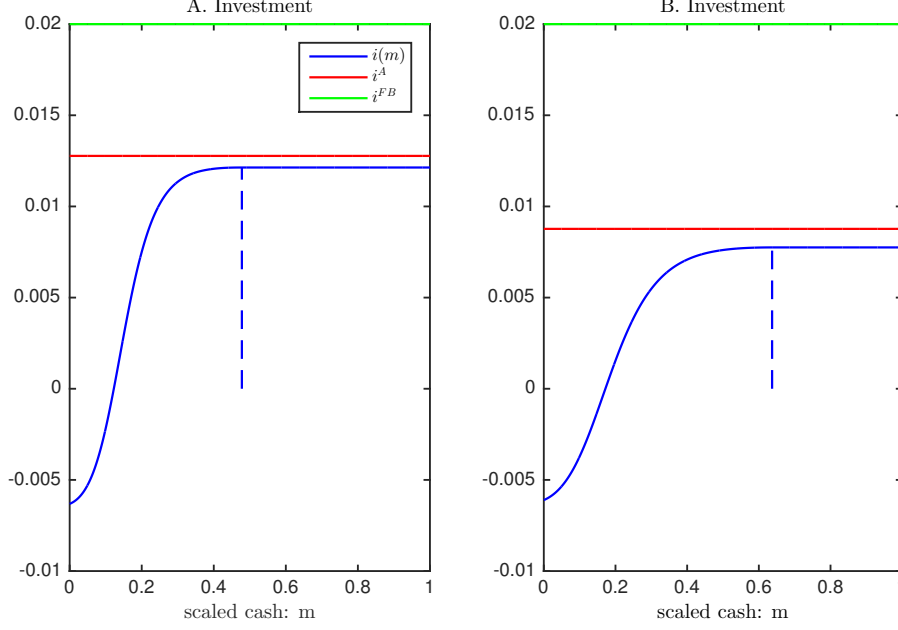


Figure 4.5: Investment with Ambiguity and Financial Frictions. This figure computes investment for three benchmarks: first best investment i^{FB} , ambiguity benchmark investment i^A , and investment for the baseline case $i(m)$. Panel A considers parameter configuration $\kappa_P = 0.05, \kappa_T = 0$. Panel B considers the parameter configuration $\kappa_P = 0, \kappa_T = 0.5$. Remaining parameter values are $\mu = 0.01, r = 0.06, \alpha = 0.17, \lambda = 0.01, \sigma_\delta = 0.25, \sigma_Y = 0.12, \omega = 0.55, \theta = 150$.

Panel A of figure (4.5) considers the case in which there is ambiguity only with respect to permanent shocks. Investment under ambiguity-aversion is lower than in the first best benchmark, i.e., $i^{FB} > i^A$, consistent with Proposition 2. Ambiguity with respect to permanent shocks reduces the expected growth rate of the firm, thereby reducing the expected benefit of investment. Moreover, if in addition to ambiguity-aversion, managers also face financial frictions, they will reduce investment further, i.e., $i^A > i(m)$. To gain intuition notice the firm's investment policy implied by equation (26) is characterized by two effects: i) investment is increasing in the marginal benefit of a unit of capital $F(m)$ (benefit of investment effect) and ii) investment is decreasing in the marginal cost of a unit of cash $F'(m)$ (cost of liquidity effect). Financial frictions increase the marginal cost of cash compared to the frictionless benchmark ($F'(m) \geq 1$), thereby reducing investment via the cost of liquidity effect. Panel B of figure (4.5) consider the complementary case when there is ambiguity only with respect to temporary shocks. In this case, investment under ambiguity aversion is lower than the benchmark case due to the reduction in the expected cash flow rate per unit of capital. The intuition for $i^A > i(m)$ carries over from the previous case.

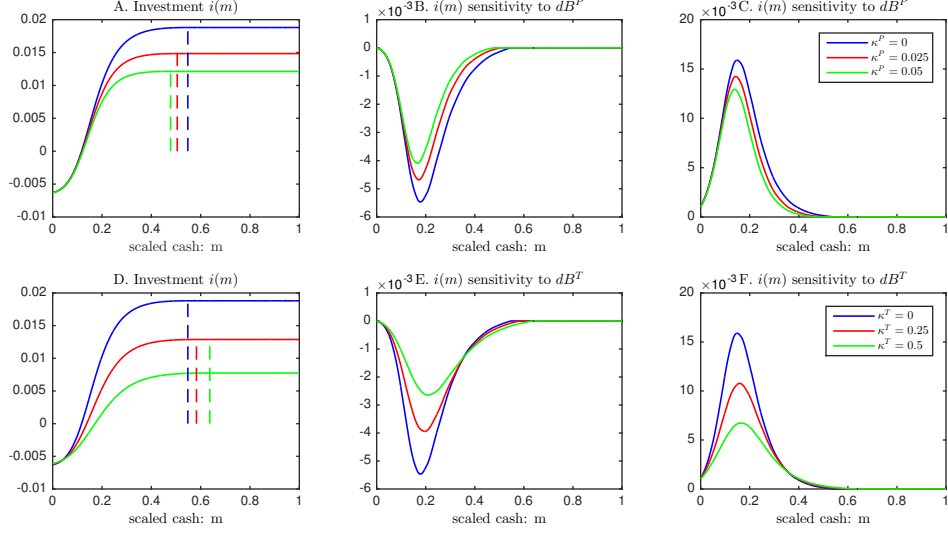


Figure 4.6: Comparative Statics for Investment and Sensitivity of Investment to Permanent and Temporary Shocks. Panels A, B, and C compute comparative statics with respect to κ_P for investment, sensitivity of investment to permanent shocks, and sensitivity of investment to temporary shocks, respectively. Panels C, D, and E compute comparative statics with respect to κ_T for investment, sensitivity of investment to permanent shocks, and sensitivity of investment to temporary shocks, respectively. Parameter values are $\mu = 0.01, r = 0.06, \alpha = 0.17, \lambda = 0.01, \sigma_\delta = 0.25, \sigma_Y = 0.12, \omega = 0.55, \theta = 150$.

We now study the effect of increasing the ambiguity-aversion parameters κ_T and κ_P on the firm's investment policy. Panels A and D of Figure (4.6) depict comparative statics of $i(m)$ with respect to κ_P and κ_T , respectively. Higher κ_P (κ_T) reduces the firm's expected growth (cash flows), thus reducing investment via the benefit of investment effect. Importantly, the cost of liquidity effect is minimal under the current assumption that at bankruptcy $F(0) = \omega q^A$. As will see in the next section, assuming there is no ambiguity after liquidation (i.e., $F(0) = \omega q^{FB}$), will activate the cost of liquidity effect. Thus, making the comparative statics with respect to κ_T and κ_P ambiguous.

Ambiguity also has implications for the sensitivity of investment to temporary and permanent shocks. We define the sensitivity of investment to permanent (temporary) shocks as the changed induced in investment by increasing the permanent (temporary) shock by one unit. We first compute the dynamics of the investment rate using Ito's formula, to obtain

$$\begin{aligned}
 di(m_t) = & \left[i'(m_t) \left((r - \lambda - \mu - i(m_t) + \sigma_\delta^2)m - g(i(m_t)) + \alpha \right) + \frac{1}{2} i''(m_t) (\sigma_Y^2 + m_t \sigma_\delta^2) \right] dt \\
 & + i'(m_t) \sigma_Y dB_t^T - i'(m_t) m \sigma_\delta dB_t^P.
 \end{aligned}$$

Now, we formally define the sensitivity of investment to permanent and temporary cash flows as

$$\langle di(m_t), dB_t^P \rangle = -i'(m_t)m\sigma_\delta dt \quad , \quad \langle di(m_t), dB_t^T \rangle = i'(m_t)\sigma_Y dt,$$

respectively.

Panels B and C of Figure (4.6) depict comparative statics of the sensitivity of investment to permanent and temporary shocks with respect to κ_P . First, we note that the sensitivity of investment to permanent shocks is negative: a permanent shock increases the firm's size δ , thereby reducing the cash to asset ratio m . Lower m implies a higher cost of liquidity, implying a lower investment rate. In contrast, the sensitivity of investment to temporary shocks is positive: a temporary shock increases the firm's scaled cash reserves m . Higher m implies a lower cost of liquidity and a higher investment rate. Second, increasing κ_T or κ_P reduces the sensitivity of investment with respect to both of the shocks. Intuitively, for low values of m the main determinant of investment is the cost of liquidity effect, while for high values of m the main determinant is the benefit of investment effect. Ambiguity reduces the benefit of investment, and thus investment increases less after an improvement in liquidity conditions.

Panels E and F of Figure (4.6) depict comparative statics of the sensitivity of investment to permanent and temporary shocks with respect to κ_T . The results and intuition for these results are similar to the previous case. Finally, we note that in contrast to the cash and dividend policy of the firm discussed in section 4.2, in which increasing κ_T and κ_P had opposing effects on the firm's policies, their effects are qualitatively similar with regards to the investment policy. Hence, our model suggests that empirical work studying the effects of ambiguity on firm behavior should focus on cash and dividend policy when separating the ambiguity with respect to permanent and temporary components.

Over-investment due to ambiguity-aversion

We conclude this section highlighting the importance of our boundary condition at liquidation: $F(0) = \omega q^A$. Instead, assume that the scrap value of the firm is not affected by ambiguity. That is, ambiguity affects the growth and cash flow rates of the firm prior to liquidation, but not after that: $F(0) = \omega q^{FB}$. Figure (4.7) depicts comparative statics of the investment rate, the scaled firm

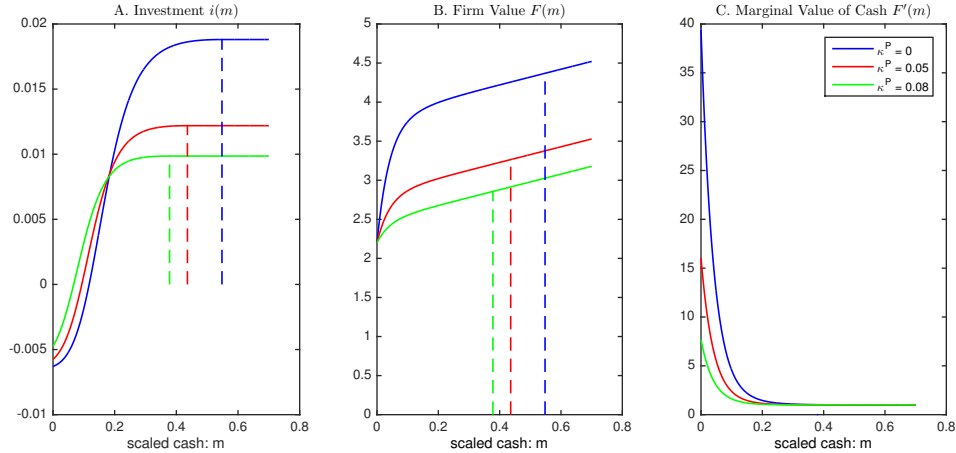


Figure 4.7: Comparative statics for investment with alternative boundary condition. Panels A, B, and C, compute comparative statics for investment, firm value, and marginal value of liquidity, with respect to κ_P . Parameter values are $\mu = 0.01, r = 0.06, \alpha = 0.17, \lambda = 0.01, \sigma_\delta = 0.25, \sigma_Y = 0.12, \omega = 0.55, \theta = 150, \kappa_T = 0$.

value, and the marginal cost of liquidity, with respect to κ_P . We notice that the effect of increasing κ_P on $i(m)$ is no longer monotonic the way it was in Figure (4.6). For low values of cash reserves an increase in ambiguity can actually *increase* investment. Recall that the optimal investment policy trades off the benefit of investment against the cost of liquidity. Panel B confirms our intuition that the benefit of investment (namely, the scaled firm value) is decreasing in the level of ambiguity. However, Panel C shows that the cost of liquidity is also decreasing in the level of ambiguity. That is, when ambiguity is higher, each additional unit of cash is less valuable, since the prospect of avoiding liquidation is less dire under the assumption that there is no ambiguity after liquidation. Thus, the benefit of investment effect and the cost of liquidity effect point in opposite directions. For high values of cash reserves m , the benefit of investment effect dominates, yielding the familiar negative relationship between ambiguity and investment. However, for low values of m , the cost of liquidity dominates, and higher ambiguity can lead to *over-investment*.¹⁹

This analysis underscores the importance of the assumptions made at default. In our model, ambiguity can lead to over-investment only when the future owners of the firm (those purchasing the firm at default) are less ambiguity-averse than the current owners of the firm. This assumption is potentially reasonable for small firms whose capital is sold to bigger and more diversified firms.

¹⁹Wu, Yang, and Zou (2017) obtained a similar result in a model using multiplier preferences, but did not discuss the critical role played by the boundary condition at liquidation.

But it may not be suitable in instances in which the firm is restructured and run by owners with similar characteristics as the original ones.²⁰

4.5 Asset Pricing Implications

In this section we explore the pricing implications of our model. In particular, we separate the ambiguity premia generated by the permanent and the temporary components. Recall the expressions for the equity premium with respect to permanent shocks $ep_P(m)$ and temporary shocks $ep_T(m)$ are given by

$$ep_P(m_t) = \frac{\sigma_\delta \kappa_P \cos \phi(m) (F(m_t) - F'(m_t)m_t)}{F(m_t)}, \quad ep_T(m_t) = \frac{\sigma_Y \kappa_T \sin \phi(m) F'(m_t)}{F(m_t)},$$

and the total equity premium $ep(m)$ is the sum of these two components:

$$ep(m_t) = ep_P(m_t) + ep_T(m_t).$$

We recall that our model generates endogenously time-varying equity premium as a function of the history of cash flows and growth rates of the firm. This history is summarized by our state variable m . We specify four effects that determine the level of equity premium for a given firm:

1. The cash-flow effect due to the lower expected cash-flow rate induced by ambiguity-aversion with respect to temporary shocks. That is, the shareholder's expected cash flow rate per unit of capital is given by $\alpha - \sigma_Y \kappa_T \sin \phi_t$. Hence, ambiguity-averse investors need to be compensated for this downward distortion in their expected cash flows.
2. The growth effect due to the lower expected growth rate induced by ambiguity-aversion with respect to permanent shocks. Shareholder's expected growth rate is given by $\mu + i - \sigma_\delta \kappa_P \cos \phi$.

Hence, ambiguity-averse investors require compensation for this downward distortion in the

²⁰We note that our results are related to the findings in Miao and Wang (2011); where they show that the optimal exercise of a real option depends on whether the payoff upon exercise is ambiguous or not. When ambiguity is only present prior to the option's exercise, ambiguity accelerates investment. However, when there is ambiguity before and after the exercise of the option ambiguity can delay investment. Our over-investment result is complementary to their since our investment technology corresponds to the neoclassical framework of incremental investment in contrast to the real options framework used in Miao and Wang (2011).

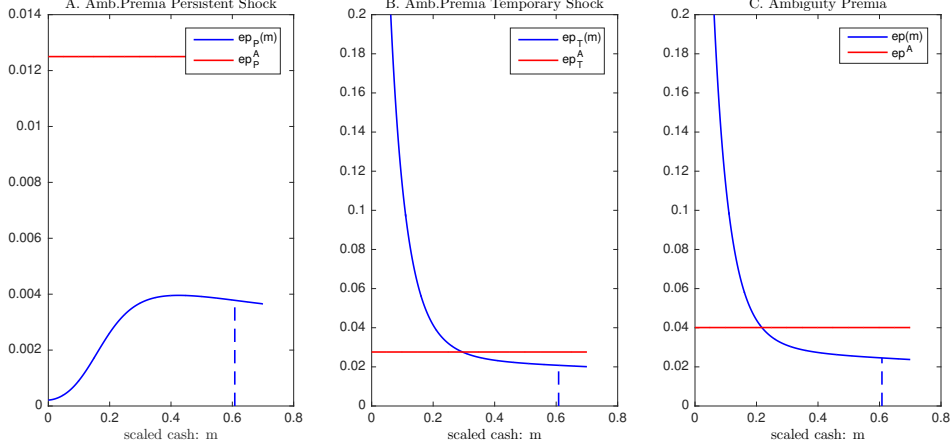


Figure 4.8: Ambiguity Premia. Panel A depicts ambiguity premia for permanent shocks. Panel B depicts ambiguity premia for temporary shocks. Panel C depicts total ambiguity premia for the firm (i.e., the sum of the permanent and temporary components). Parameter values are $\mu = 0.01, r = 0.06, \alpha = 0.17, \lambda = 0.01, \sigma_\delta = 0.25, \sigma_Y = 0.12, \omega = 0.55, \theta = 150, \kappa_T = 0.5, \kappa_P = 0.05$.

growth rate.

3. The inefficient liquidation effect induced by the distortion in the likelihood that the firm will be liquidated after running out of cash $m = 0$. Ambiguity-aversion distorts the dynamics of m , and thus the expected probability of liquidation. Since liquidation is assumed to be inefficient, ambiguity-averse investors require compensation for this potential loss.
4. The composition effect induced by the fact that financial frictions force companies to hold cash. Thus, the cash flows accruing to shareholders correspond to the sum of the cash flows from the firm's asset plus the interest from the cash reserves. Because the interest in cash is assumed to be risk-less (and unambiguous), a higher proportion of cash will reduce the firm's total equity premium.

Figure (4.8) depicts $ep_P(m)$, $ep_T(m)$, and $ep(m)$ for the case in which the firm faces ambiguity-aversion with respect to both permanent and temporary shocks, and compares them to their counterpart in the absence of financial frictions (i.e., the ambiguity-aversion baseline model ep_P^A , ep_T^A , and ep^A). Panel A shows that $ep_P(m) < ep_P^A$. Two forces explain this: First, as discussed in section 4.1, financial frictions induce more ambiguity with respect to short-term shocks than to long-term shocks. Therefore, the growth effect induces a lower expected growth rate when the firm does not have financial frictions. Second, financial frictions induce precautionary cash holdings, thus

the composition effect reduces the associated equity premia. Moreover, $ep_P(m)$ is non-monotonic in the level of cash holdings. Importantly, financially distressed firms with low m and close to liquidation have a low equity premium. To gain intuition, consider an investor who is ambiguity-averse with respect to the permanent (growth) shocks. Her expected growth rate is distorted downwards, which implies the ratio of cash to assets is expected to increase. Thus, she perceives liquidation as a *less* likely outcome, thereby requiring a smaller compensation for holding this asset. This inefficient liquidation effect is only significant when the firm is financially distressed (when m is low). For intermediate levels of m , the growth rate effect makes the investor pessimistic about the firm's growth prospects, and she needs to be compensated for it. Finally, for high levels of m the composition effect kicks in and lowers the equity premia.

Panel B shows that $ep_T(m)$ is decreasing in m and that the ambiguity premia associated with temporary shocks at the payout boundary \bar{m} is lower in the baseline model than in the ambiguity benchmark (i.e., $ep_T(\bar{m}) < ep_T^A$). Ambiguity aversion distorts downwards the expected cash flows of the firm. For a firm with low cash reserves this distortion activates the inefficient liquidation effect, by increasing the perceived probability of liquidation. Thus, the ambiguity premium shoots up as m gets close to 0. As m grows, inefficient liquidation becomes less of a concern, and instead the composition effects renders the firm's expected returns less ambiguous. Because the composition effect is not present in the ambiguity benchmark ep_T^A (no need to hold cash in the absence of financial frictions), it is not surprising that $ep_T^A > ep_T(\bar{m})$. Panel C depicts the total equity premium, and it shows that for our particular calibration of the ambiguity-aversion parameters ($\kappa_T = 0.5$ and $\kappa_P = 0.05$), the temporary effect component dominates, implying a decreasing relationship between $ep(m)$ and m .

In conclusion, our model implies a different relationship between the cost of financial distress and the implied ambiguity premia when the investor faces ambiguity with respect to permanent versus temporary shocks.

Comparative Statics

In Figure (4.9) we compute comparative statics of the firm's equity premium with respect to the recovery parameter ω . Lower ω implies the fraction lost at liquidation is greater, implying a higher cost of financial distress. Panel A considers the case in which investors only face ambiguity with

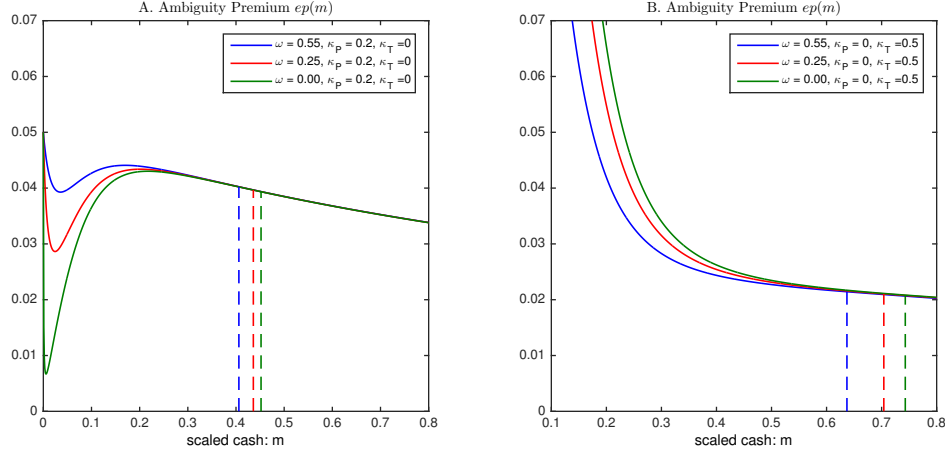


Figure 4.9: Ambiguity Premia. Panel A computes comparative statics for ambiguity premia with respect to κ_P . Panel B computes comparative statics for ambiguity premia with respect to κ_T . Parameter values are $\mu = 0.01, r = 0.06, \alpha = 0.17, \lambda = 0.01, \sigma_\delta = 0.25, \sigma_Y = 0.12, \omega = 0.55, \theta = 150$.

respect to the permanent shocks (in particular we set $\kappa_P = 0.2$ and $\kappa_T = 0$). Surprisingly, higher cost of financial distress imply a *lower* equity premium. Higher cost of financial distress implies that liquidation is more inefficient, which amplifies the role of the inefficient liquidation effect. This effect reduces equity premium when the investor is ambiguity-averse with respect to permanent shocks, because the investor expects δ to grow slowly, and $m = M/\delta$ to grow faster; rendering liquidation less likely.

Panel B considers the case in which investors only face ambiguity with respect to the temporary shocks (in particular we set $\kappa_T = 0.5$ and $\kappa_P = 0$). In contrast to the previous case, higher cost of financial distress implies a higher equity premium. As the firm gets close to liquidation, investors become pessimistic about the firm's future cash flows and expect inefficient liquidation to occur with a higher probability. Thus, investors require a higher compensation for holding the asset when the cost of inefficient liquidation is higher (i.e., when the recovery value ω is lower).

Importantly, our model can reconcile the empirical regularity that financially distressed firms (i.e., firms close to liquidation) command a low equity premium, as documented by Dichev (1998); Campbell, Hilscher, and Szilagyi (2008); and Hackbarth, Haselmann, and Schoenherr (2015). Moreover, according to our model, such empirical regularity should be concentrated among firms that face most ambiguity about growth prospects (i.e., high κ_P and low κ_T), and for which liquidation is very inefficient (i.e., low ω). Garlappi and Yang (2011) show that the distress risk puzzle is stronger for firms with larger asset bases, low R&D spending, and high industry sales concentra-

tion. We conjecture that because firms with large asset bases are typically older, have longer track records of their history of cash flows, and face less ambiguity about short-term shocks (i.e., have low κ_T). Also, firms with high κ_P are less likely to invest in R&D and more in capital expenditures. Finally, as argued by Shleifer and Vishny (1992) when a firm’s assets are specific to a particular industry (proxied by high industry concentration in Garlappi and Yang, 2011), they are subject to a substantial fire-sale discount in liquidation auctions (i.e., low ω).

In conclusion our model seems broadly consistent with the evidence documented in the empirical asset pricing literature. However, we have made use of our economic intuition to link ambiguity with various firm characteristics when relating the implications of our model with existing empirical findings. Ideally, we would like to directly test the model’s predictions on the data, but the lack of a universally acceptable way of measuring ambiguity in the data makes this a very challenging task.²¹

5 Conclusions

We analyze a situation in which the ambiguity-averse manager of an all-equity financed firm is unsure about the distribution of the growth and cash flow shocks affecting its firm. In order to protect himself against this uncertainty, he decides to maximize the firm’s net present value in the worst-case scenario plausible. We characterize the manager’s beliefs, and the optimal dividend, equity issuance, and investment policies subject to the constraint of keeping the firm’s cash reserves positive at all times.

In our setting the manager’s worst-case beliefs are time varying, and depend on the history of shocks. Importantly, we show that the manager forms beliefs (and therefore acts) “as if” he displayed extrapolation bias. That is, extrapolation bias and ambiguity aversion are observationally equivalent from the perspective of an outside observer having access to the firm’s policies and the manager’s expectations about future performance. Moreover, this extrapolation bias is exhibited with respect to the two types of shocks the firm is exposed to: permanent (growth) and temporary (cash flow) shocks.

Furthermore, our continuous time model allows us to draw sharp characterizations of the rel-

²¹For a recent proposed empirical measure of ambiguity see Izhakian (2015), and for an empirical exploration of ambiguity on R&D expenditure see Coiculescu, Izhakian, and Ravid (2018).

evant comparative statics. In particular, we show increasing ambiguity of permanent shocks has the opposite effect of increasing volatility of permanent shocks on the firm's dividend and equity issuance policies. Thus, underscoring the potential of ambiguity-aversion to expand the range of behavior plausible with purely risky shocks.

Finally, our paper has very different asset pricing implications for the equity premia associated with permanent and temporary shocks. While equity premia of temporary shocks is increasing in the degree of financial distress, the equity premia of permanent shocks is hump-shaped in financial distress. As a result, our paper can rationalize the empirical regularity that financially distressed firms command low equity premia (i.e., the distress puzzle).

Our results raise several interesting questions for future research. For example, under which general conditions is there a link between extrapolation bias and ambiguity-aversion? Is there a way of empirically distinguishing whether agents exhibit an "irrational" extrapolation bias or are they rationally protecting themselves against unknown unknowns? If shareholders, bondholders, and managers have varying degrees of ambiguity-aversion, which kind of capital structure and optimal policies would be optimal? These and other questions are the subject of ongoing research.

References

- [1] Anderson, Evan W., Lars Peter Hansen, and Thomas J. Sargent, 2003, A quartet of semigroups for model specification, robustness, prices of risk, and model detection, *Journal of the European Economic Association* 1.1: 68-123.
- [2] Breuer, Wolfgang, Marc O. Rieger, and K. Can Soypak, 2016, Corporate cash holdings and ambiguity aversion, *Review of Finance* 21.5: 1933-1974.
- [3] Bolton, P., H. Chen, and N. Wang, 2011, A Unified Theory of Tobin's Q, Corporate Investment, Financing, and Risk Management, *Journal of Finance*, 66, 1545-1578.
- [4] Bolton, P., H. Chen, and N. Wang. 2013, Market Timing, Investment, and Risk Management, *Journal of Financial Economics*, 109, 40-62
- [5] Byun, Seong K., Valery Polkovnichenko, and Michael J. Rebell, 2018, Composition of Cash Flow Shocks and Debt Financing, Working Paper.
- [6] Chen, Zengjing, and Larry Epstein, 2002, Ambiguity, risk, and asset returns in continuous time, *Econometrica* 70.4: 1403-1443.
- [7] Campbell, John Y., Jens Hilscher, and Jan Szilagyi, In search of distress risk, *The Journal of Finance* 63.6: 2899-2939.
- [8] Décamps, J.-P., S. Gryglewicz, E. Morellec, and S. Villeneuve. 2017, Corporate Policies with Permanent and Transitory Shocks, *Review of Financial Studies*, forthcoming.
- [9] Décamps, J., T. Mariotti, J.C. Rochet, and S. Villeneuve, 2011, Free Cash Flow, Issuance Costs, and Stock Prices, *Journal of Finance*, 1501-1544.
- [10] DeMarzo, P., M. Fishman, Z. He and N. Wang, 2012, Dynamic Agency and the Q Theory of Investment, *Journal of Finance*, 67, 2295-2340.
- [11] Dicks, David, and Paolo Fulghieri, 2018, Uncertainty aversion and systemic risk, *Journal of Political Economy*, forthcoming.
- [12] Dicks, David, and Paolo Fulghieri, 2014, Ambiguity, disagreement, and corporate control, Working Paper.
- [13] Dicks, David, and Paolo Fulghieri, 2017, Innovation waves, investor sentiment, and mergers, Working Paper.
- [14] Epstein, Larry G., and Shaolin Ji, 2013, Ambiguous volatility and asset pricing in continuous time, *The Review of Financial Studies* 26.7: 1740-1786.
- [15] Epstein, Larry G., and Shaolin Ji, 2014, Ambiguous volatility, possibility and utility in continuous time, *Journal of Mathematical Economics* 50: 269-282.
- [16] Epstein, Larry G., and Martin Schneider, 2003, Recursive multiple-priors, *Journal of Economic Theory* 113.1: 1-31.

- [17] Epstein, Larry G., and Martin Schneider, 2010, Ambiguity and asset markets, *Annual Review of Financial Economics*
- [18] Fuster, Andreas, David Laibson, and Brock Mendel, 2010, Natural expectations and macroeconomic fluctuations, *Journal of Economic Perspectives* 24.4 : 67-84.
- [19] Garlappi, Lorenzo, Ron Giammarino, and Ali Lazrak, 2017, Ambiguity and the corporation: Group disagreement and underinvestment, *Journal of Financial Economics* 125.3: 417-433.
- [20] Garlappi, Lorenzo, and Hong Yan, 2011, Financial distress and the cross-section of equity returns, *The Journal of Finance* 66.3: 789-822.
- [21] Gorbenko, A. and I. Strebulaev, 2010, Temporary versus Permanent Shocks: Explaining Corporate Financial Policies, *Review of Financial Studies*, 23, 2591-2647.
- [22] Gryglewicz, Sebastian, Simon Mayer, and Erwan Morellec, 2018, Agency conflicts over the short and long run: Short-termism, long-termism, and pay-for-luck, Working Paper.
- [23] Gryglewicz, Sebastian, Lorian Mancini, Erwan Morellec, Enrique J. Schroth, and Philip Valta, 2017, Transitory versus permanent shocks: Explaining corporate savings and investment, Working Paper.
- [24] Hayashi, Fumio, 1982, Tobin's marginal q and average q: A neoclassical interpretation, *Econometrica*: 213-224.
- [25] Hackbarth, Dirk, Rainer Haselmann, and David Schoenherr, 2015, Financial distress, stock returns, and the 1978 Bankruptcy Reform Act, *The Review of Financial Studies* 28.6 (2015): 1810-1847.
- [26] Hoberg, Gerard, and Gordon Phillips, 2010, Real and financial industry booms and busts, *The Journal of Finance* 65.1: 45-86.
- [27] Izhakian, Yehuda, 2017, Expected utility with uncertain probabilities theory, *Journal of Mathematical Economics* 69: 91-103.
- [28] Izhakian, Yehuda, David Yermack, and Jaime Zender, 2016, Ambiguity and the Tradeoff Theory of Capital Structure, Working Paper.
- [29] Jensen, Michael C, 1986, Agency costs of free cash flow, corporate finance, and takeovers, *American Economic Review* 76.2: 323-329.
- [30] Lee, Seokwoo, 2014, Knightian uncertainty and capital structure: Theory and evidence, working paper.
- [31] Leland, H., 1994, Corporate Debt Value, Bond Covenants, and Optimal Capital Structure, *Journal of Finance*, 49, 1213-1252.
- [32] Malenko, Andrey, and Anton Tsoy, 2018, Asymmetric Information and Security Design under Knightian Uncertainty, Working Paper.
- [33] Malmendier, Ulrike, and Stefan Nagel, Learning from inflation experiences, *The Quarterly Journal of Economics* 131.1: 53-87.

- [34] Miao, Jianjun, and Alejandro Rivera, 2016, Robust contracts in continuous time, *Econometrica* 84.4 (2016): 1405-1440.
- [35] Miao, Jianjun, and Neng Wang, Risk, 2011, uncertainty, and option exercise, *Journal of Economic Dynamics and Control* 35.4: 442-461.
- [36] Moreno-Bromberg, Santiago, and Jean-Charles Rochet, 2018, Continuous-Time Models in Corporate Finance, Banking, and Insurance: A User's Guide. Princeton University Press, 2018.
- [37] Paaso, Mikael, 2018, Extrapolation and Real Investment, Working Paper.
- [38] Shin, Donghwa, 2018, Extrapolation and Complexity, Working Paper.
- [39] Strebulaev, I., and T. Whited, 2012, Dynamic Models and Structural Estimation in Corporate Finance, *Foundations and Trends in Finance*, 6, 1-163.
- [40] Strzalecki, Tomasz, Axiomatic foundations of multiplier preferences, *Econometrica* 79.1: 47-73.
- [41] Wu, Yaoyao, Jinqiang Yang, and Zhentao Zou, 2017, Dynamic corporate investment and liquidity management under model uncertainty, *Economics Letters* 155: 9-13.

Appendix A

Proof of Proposition 1:

Proof. The HJB equation for the First Best Benchmark is given by

$$rV^{FB}(\delta) = \max_i \left\{ \delta(\alpha - g(i)) + V_\delta^{FB}(\delta)\delta(\mu + i) + \frac{1}{2}V_{\delta\delta}^{FB}(\delta, M)\sigma_\delta^2\delta^2 \right\}.$$

We conjecture the solution for this problem has the form $V^{FB}(\delta) = q^{FB}\delta$, thus

$$rq^{FB} = \max_i \left\{ (\alpha - g(i)) + q^{FB}(\mu + i) \right\} \quad (33)$$

and the FOC and SOC for i are given by:

$$q^{FB} - (1 + \theta i^{FB}) = 0, \quad (34)$$

$$-\theta \leq 0.$$

Equations (33) and (34), plus the transversality condition $\lim_{t \rightarrow \infty} E[e^{-rt}\delta_t] = 0$, yield a unique solution for i^{FB} and q^{FB}

$$i^{FB} = \frac{(\theta(-\theta\mu^2 + 2\theta\mu r + 2\mu - \theta r^2 - 2r + 2\alpha))^{1/2} - \mu\theta + r\theta}{\theta},$$

$$q^{FB} = 1 - (-\theta(-\theta\mu^2 + 2\theta\mu r + 2\mu - \theta r^2 - 2r + 2\alpha))^{1/2} - \mu\theta + r\theta.$$

A standard verification argument shows the solution to the HJB coincides with the value function for this problem. \square

Proof of Proposition 2:

The HJB equation for the Ambiguity Benchmark is given by

$$rV^A(\delta) = \max_i \min_{\phi \in [0, 2\pi]} \left\{ \delta(\alpha - g(i)) - \delta\sigma_Y\kappa_T \sin \phi + V_\delta^A(\delta)\delta(\mu + i - \kappa_P\sigma_\delta \cos \phi) + \frac{1}{2}V_{\delta\delta}^A(\delta, M)\sigma_\delta^2\delta^2 \right\}.$$

Substitute into this equation our guess for the form of the solution $V^A(\delta) = q^A\delta$ to obtain:

$$rq^A = \max_i \min_{\phi \in [0, 2\pi]} \left\{ (\alpha - g(i)) - \sigma_Y\kappa_T \sin \phi + q^A(\mu + i - \kappa_P\sigma_\delta \cos \phi) \right\}, \quad (35)$$

where the FOC for i and ϕ are given by

$$q^A - (1 + \theta i^A) = 0, \quad (36)$$

$$\sigma_Y\kappa_T \cos \phi^A - q^A\sigma_\delta\kappa_P \sin \phi^A = 0, \quad (37)$$

respectively, and the SOC are given by

$$-\theta \leq 0,$$

$$-\sigma_Y\kappa_T \sin \phi^A - q^A\sigma_\delta\kappa_P \cos \phi^A \geq 0 \text{ if } \phi^A \in [\pi, \frac{3}{2}\pi].$$

Substituting back the optimizers i^A and ϕ^A into (35) yields

$$rq^A = (\alpha - g(i^A)) - \sigma_Y \kappa_T \sin \phi^A + q^A (\mu + i^A - \kappa_P \sigma_\delta \cos \phi^A), \quad (38)$$

which together with (36), (37), and the transversality condition $\lim_{t \rightarrow \infty} E[e^{-rt} \delta_t] = 0$ determine the firm value and the optimal controls. A standard verification theorem shows the solution to the HJB coincides with the value function for this problem.

Auxiliary Lemmas for Verification

The following two auxiliary lemmas will be useful in the verification result. First, we show that it is without loss of generality that the minimization over the density generator $h = (h^P, h^T)$ takes place at the boundary of the set $H(\cdot) = \left\{ y \in R^2 : \left| \frac{y_1}{\kappa_P} \right|^\gamma + \left| \frac{y_2}{\kappa_T} \right|^\gamma \leq 1 \right\}$

Lemma 4. $F(m)$ is a solution to (25) if and only if it is a solution to:

$$rF(m) = \max_i \min_{h \in H(\cdot)} \left\{ \begin{array}{l} (F(m) - F'(m)m)(\mu + i + h^P) \\ + F'(m)(\alpha - g(i) + h^T + (r - \lambda)m) + \frac{1}{2} F''(m)(\sigma_Y^2 + m^2 \sigma_\delta^2) \end{array} \right\}$$

Proof. Suppose for a contradiction that a minimizer is achieved at the interior of the set $H(\cdot)$, i.e.,

$$(h^{P*}, h^{T*}) \in \arg \min_{h \in H(\cdot)} \left\{ (F(m) - F'(m)m) h^P + F'(m) h^T \right\}$$

$$(h^{P*}, h^{T*}) \in \text{int}(H(\cdot)).$$

A necessary condition for an interior minimizer is that the gradient of the objective function be equal to zero:

$$(F(m) - F'(m)m) = 0 \quad \text{and} \quad F'(m) = 0$$

which contradicts the fact that $F'(m) \geq 1$ for every m .

Second, the following lemma will help us show it is optimal to issue equity only at $m = 0$.

□

Lemma 5. If $F(m)$ is a concave increasing function that satisfies

$$F(0) = \max \left\{ \max_m (F(m) - \rho_1(m + \rho_0)); \omega q^A \right\}. \quad (39)$$

then

$$F(m) \geq F\left(m + \frac{i}{\rho_1} - \rho_0\right) - i \quad (40)$$

for all $m \geq 0, i \geq 0$.

Proof. Suppose for a contradiction that (40) does not hold. That is, there exist \tilde{m} and \tilde{i} such that

$$F(\tilde{m}) < F\left(\tilde{m} + \frac{\tilde{i}}{\rho_1} - \rho_0\right) - \tilde{i}. \quad (41)$$

However, from (39) it follows that

$$F(0) \geq F(m) - \rho_1(m + \rho_0) \quad (42)$$

for every $m \geq 0$. Substituting $m = \tilde{m} + \frac{\tilde{i}}{\rho_1} - \rho_0$ into (42) yields

$$F(0) \geq F(\tilde{m} + \frac{\tilde{i}}{\rho_1} - \rho_0) - \rho_1\tilde{m} - \tilde{i} \quad (43)$$

Combining (43) and (41) it obtains that

$$F(\tilde{m}) \leq F(\tilde{m} + \frac{\tilde{i}}{\rho_1} - \rho_0) - \tilde{i} \leq F(0) + \rho_1\tilde{m} \quad (44)$$

which implies that

$$\frac{F(\tilde{m}) - F(0)}{\tilde{m}} \leq \rho_1,$$

and since F is concave then $F'(m) < \rho_1$ for $\tilde{m} < m$. Using a Taylor expansion of F around \tilde{m} implies that there exist $x \geq \tilde{m}$ such that

$$F(\tilde{m} + \frac{\tilde{i}}{\rho_1} - \rho_0) = F(\tilde{m}) + \underbrace{F'(x)}_{\leq \rho_1} (\frac{\tilde{i}}{\rho_1} - \rho_0) \leq F(\tilde{m}) + \tilde{i} - \rho_1\rho_0$$

which combined with (44) implies that

$$F(\tilde{m}) \leq F(\tilde{m}) - \rho_1\rho_0$$

which is a contradiction. □

Verification Theorem

We proceed in two steps. Step 1, shows that problem (14) can be re-written as a one-dimensional control problem. Step 2 shows that the solution to the the variational system (25)-(31) coincides with the solution of the one dimensional control problem and derives the optimal dividend, issuance, investment policy, and belief distortion. To avoid confusion, we denote V^* and F^* the value functions of the stochastic control problems and V and F denote the solution to variational systems.

Step 1: Fix an admissible density generator h and associated probability measure $\mathbb{Q}^h \in \mathcal{P}^H$, and define the probability measure $\tilde{\mathbb{Q}}^h$ by

$$\left(\frac{d\mathbb{Q}^h}{d\tilde{\mathbb{Q}}^h} \right) = Z_t = \exp \left\{ -\frac{1}{2}\sigma_A^2 t + \sigma_A W_t^P \right\}, \quad \forall t \geq 0, \quad (45)$$

on (Ω, \mathcal{F}) . By Girsanov's Theorem, (\tilde{W}_t^P, W_t^T) with $\tilde{W}_t^P = -\sigma_A t + W_t^P$ is a two dimensional Brownian motion process under the probability measure $\tilde{\mathbb{Q}}^h$. We have

Proposition 6. *The value function V^* of problem (14) satisfies:*

$$V^*(\delta, M) = \delta F^*\left(\frac{M}{\delta}\right),$$

the function F^* is defined on $[0, \infty)$ by

$$F^*(m) = \max_{\tau_n, e_n, L, i} \min_h f(m; \tau_n, e_n, L, i, h) \quad (46)$$

with

$$f(m; \tau_n, e_n, L, i, h) = E_c^{\tilde{\mathbb{Q}}^h} \left[\int_0^{\tau_0} e^{-rt + \int_0^t (\mu + i_s + \sigma_\delta h_s^P) ds} (d\tilde{L}_t - d\tilde{E}_t) + e^{-r\tau_0 + \int_0^{\tau_0} (\mu + i_s + \sigma_\delta h_s^P) ds} \omega q^{FB} \right] \quad (47)$$

and

$$\begin{aligned} m_0 = m, \quad dm_t &= (\alpha - g(i_t) + \sigma_Y h_t^T + m_t(r - \lambda - \mu - i_t - \sigma_\delta h_t^P)) dt \\ &+ \sqrt{\sigma_Y^2 + \sigma_\delta^2 m_t^2} dW_t^C + \frac{d\hat{E}_t}{\rho_1} - d\hat{\Phi}_t - d\hat{L}_t, \end{aligned} \quad (48)$$

where W^m is Brownian motion under $\tilde{\mathbb{Q}}^h$ and

$$\tau_0 = \inf\{t \geq 0 : m_t = 0\}, \quad (49)$$

$$\hat{\Phi}_t = \sum_{n \geq 1} \rho_0 1_{\{\tau_n \leq t\}}, \quad (50)$$

$$\hat{E}_t = \sum_{n \geq 1} \hat{e}_n 1_{\{\tau_n < t\}} \text{ with } \hat{e}_n = e_n \delta \tau_n, \quad (51)$$

$$\hat{L}_t = \int_0^t \frac{1}{\delta_s} dL_s. \quad (52)$$

Proof. Fix an arbitrary policy (τ_n, e_n, L, i) . Applying Ito's formula to $(e^{-r(t \wedge \tau_0)} M_{t \wedge \tau_0})$ and letting t go to ∞ yields

$$\begin{aligned} E^{\mathbb{Q}^h} \left[\int_0^{\tau_0} e^{-rt} (dL_t - dE_t) \right] &= M_0 + E^{\mathbb{Q}^h} \left[\int_0^{\tau_0} e^{-rt} \left(-\lambda M_t + \delta_t (\alpha - g(i_t) + \sigma_Y h_t^T) \right) dt \right] \\ &- E^{\mathbb{Q}^h} \left[\int_0^{\tau_0} e^{-rt} \left(\frac{\rho_1 - 1}{\rho_1} dE_t + d\hat{\Phi}_t \right) \right], \end{aligned} \quad (53)$$

which can be rewritten as

$$\begin{aligned} \frac{1}{\delta_0} E^{\mathbb{Q}^h} \left[\int_0^{\tau_0} e^{-rt} (dL_t - dE_t) \right] &= \frac{M_0}{\delta_0} + E^{\mathbb{Q}^h} \left[\int_0^{\tau_0} e^{-rt} \frac{Z_t}{\delta_t} e^{\int_0^t (\mu + i_s + \sigma_\delta h_s^P) ds} \left(-\lambda M_t + \delta_t (\alpha - g(i_t) + \sigma_Y h_t^T) \right) dt \right] \\ &- E^{\mathbb{Q}^h} \left[\int_0^{\tau_0} e^{-rt} \frac{Z_t}{\delta_t} e^{\int_0^t (\mu + i_s + \sigma_\delta h_s^P) ds} \left(\frac{\rho_1 - 1}{\rho_1} dE_t + d\hat{\Phi}_t \right) \right], \end{aligned} \quad (54)$$

noting that $\frac{Z_t}{\delta_t} = \frac{1}{\delta_0} e^{-\int_0^t (\mu + i_s + \sigma_\delta h_s^P) ds}$. The change in probability (45) yields

$$\begin{aligned} \frac{1}{\delta_0} E^{\mathbb{Q}^h} \left[\int_0^{\tau_0} e^{-rt} (dL_t - dE_t) \right] &= \frac{M_0}{\delta_0} + E^{\tilde{\mathbb{Q}}^h} \left[\int_0^{\tau_0} e^{-rt + \int_0^t (\mu + i_s + \sigma_\delta h_s^P) ds} \left(-\lambda \frac{M_t}{\delta_t} + (\alpha - g(i_t) + \sigma_Y h_t^T) \right) dt \right] \\ &- E^{\tilde{\mathbb{Q}}^h} \left[\int_0^{\tau_0} e^{-rt + \int_0^t (\mu + i_s + \sigma_\delta h_s^P) ds} \left(\frac{\rho_1 - 1}{\rho_1} d\hat{E}_t + d\hat{\Phi}_t \right) \right]. \end{aligned} \quad (55)$$

Now apply Ito's formula to $\frac{M_t}{\delta_t}$, to obtain

$$\begin{aligned} d\left(\frac{M_t}{\delta_t}\right) &= \left(\alpha - g(i_t) + \sigma_Y h_t^T + \frac{M_t}{\delta_t}(r - \lambda - \mu - i_t - \sigma_\delta h_t^P)\right) dt \\ &\quad + \sigma_Y dW_t^T - \frac{M_t}{\delta_t} \sigma_\delta d\tilde{W}_t^P + \frac{1}{\delta} \left(\frac{dE_t}{\rho_1} - dL_t - d\Phi_t\right) \end{aligned} \quad (56)$$

or equivalently,

$$\begin{aligned} d\left(\frac{M_t}{\delta_t}\right) &= \left(\alpha - g(i_t) + \sigma_Y h_t^T + \frac{M_t}{\delta_t}(r - \lambda - \mu - i_t - \sigma_\delta h_t^P)\right) dt \\ &\quad + \sqrt{\sigma_Y^2 + \sigma_\delta^2 \frac{M_t^2}{\delta_t^2}} dW_t^m + \frac{1}{\delta} \left(\frac{dE_t}{\rho_1} - dL_t - d\Phi_t\right), \end{aligned} \quad (57)$$

where W^m is a Brownian motion under $\tilde{\mathbb{Q}}^h$. Finally, we apply Ito's formula to $e^{-rt + \int_0^t (\mu + i_s + \sigma_\delta h_s^P) ds} \frac{M_t \wedge \tau_0}{\delta_t \wedge \tau_0}$ and letting t go to ∞ , we obtain:

$$\begin{aligned} E^{\tilde{\mathbb{Q}}^h} \left[\int_0^{\tau_0} e^{-rt + \int_0^t (\mu + i_s + \sigma_\delta h_s^P) ds} (d\hat{L}_t - d\hat{E}_t) \right] &= \\ \frac{M_0}{\delta_0} + E^{\tilde{\mathbb{Q}}^h} \left[\int_0^{\tau_0} e^{-rt + \int_0^t (\mu + i_s + \sigma_\delta h_s^P) ds} \left[\left(-\lambda \frac{M_t}{\delta_t} + (\alpha - g(i_t) + \sigma_Y h_t^T)\right) dt \right] \right] \\ + E^{\tilde{\mathbb{Q}}^h} \left[\int_0^{\tau_0} e^{-rt + \int_0^t (\mu + i_s + \sigma_\delta h_s^P) ds} \left[-d\hat{\Phi}_t - \frac{\rho_1 - 1}{\rho_1} d\hat{E}_t \right] \right]. \end{aligned} \quad (58)$$

Combining (58) and (55) and noting that $E^{\mathbb{Q}^h} [e^{-r\tau_0} \omega q^{FB} \delta_{\tau_0}] = \delta_0 E^{\tilde{\mathbb{Q}}^h} [e^{-r\tau_0 + \int_0^{\tau_0} (\mu + i_s + \sigma_\delta h_s^P) ds} \omega q^{FB}]$ we obtain that

$$\begin{aligned} &E^{\mathbb{Q}^h} \left[\int_0^{\tau_0} e^{-rt} (dL_t - dE_t) + e^{-r\tau_0} \omega q^{FB} \delta_{\tau_0} \right] \\ &= \delta_0 E^{\tilde{\mathbb{Q}}^h} \left[\int_0^{\tau_0} e^{-rt + \int_0^t (\mu + i_s + \sigma_\delta h_s^P) ds} (d\hat{L}_t - d\hat{E}_t) + e^{-r\tau_0 + \int_0^{\tau_0} (\mu + i_s + \sigma_\delta h_s^P) ds} \omega q^{FB} \right]. \end{aligned}$$

To conclude we maximize over the admissible policies, and minimize over the probability measures and obtain that the problem

$$\max_{\tau_n, e_n, L, i} \min_h \delta_0 E^{\tilde{\mathbb{Q}}^h} \left[\int_0^{\tau_0} e^{-rt + \int_0^t (\mu + i_s + \sigma_\delta h_s^P) ds} (d\hat{L}_t - d\hat{E}_t) + e^{-r\tau_0 + \int_0^{\tau_0} (\mu + i_s + \sigma_\delta h_s^P) ds} \omega q^{FB} \right]$$

is equivalent to (46)-(52). □

Step 2: We show that the value function for the one dimensional control problem F^* and the solution to the variational system F coincide. We prove this verification theorem in two steps. First, we show F is an upper bound for F^* . Second, we show that they are equal. The following two lemmas will be useful to accomplish these two steps.

Lemma 7. *Suppose there exists a increasing concave twice continuously differentiable solution $F(m)$ to the variational system (25)-(31). For any admissible policy $\Gamma = (\tau_n, e_n, L, i)$, and a belief distortion $h^* = (h_t^P, h_t^T)$ given by (27), then*

$$F(m) \geq f(m; \tau_n, e_n, L, i, h^*) \quad (59)$$

Proof. Consider the gain process $G_{t \wedge \tau_0}^{\Gamma, h^*}$ defined by

$$\begin{aligned} G_{t \wedge \tau_0}^{\Gamma, h^*} &= \int_0^{t \wedge \tau_0} e^{-rs + \int_0^s (\mu + i_u + \sigma_\delta h_u^P) du} (d\hat{L}_s - d\hat{E}_s) + 1_{\{t < \tau_0\}} e^{-rt + \int_0^t (\mu + i_u + \sigma_\delta h_u^P) du} F(m_t) \\ &\quad + 1_{\{t \geq \tau_0\}} e^{-r\tau_0 + \int_0^{\tau_0} (\mu + i_u + \sigma_\delta h_u^P) du} \omega q^{FB}. \end{aligned}$$

We will show that $G_{t \wedge \tau_0}^{\Gamma, h^*}$ is a $\mathbb{Q}^{\tilde{h}^*}$ super-martingale. Using Ito's formula yields

$$\begin{aligned} e^{rt - \int_0^t (\mu + i_u + \sigma_\delta h_u^P) du} dG_{t \wedge \tau_0}^{\Gamma, h^*} &= (d\hat{L}_t - d\hat{E}_t) + F'(m_t) (\alpha - g(i) - \sigma_Y \kappa_T \sin \phi + (r - \lambda)m) dt \\ &\quad + \frac{1}{2} F''(m_t) (\sigma_Y^2 + m_t^2 \sigma_\delta^2) dt + F(m_t) (-r + \mu + i_t - \sigma_\delta \kappa_P \cos \phi) dt + F(m_t) \sqrt{\sigma_Y^2 + m_t^2 \sigma_\delta^2} dW_t^m \\ &\quad + F'(m_t) (-d\hat{L}_t) + 1_{\{t = \tau_k\}} \left(F \left(m_t + \frac{e_{\tau_k}}{\rho_1} - \rho_0 \right) - F(m_t) \right) \end{aligned}$$

taking expectations we obtain that

$$\begin{aligned} E_t^{\mathbb{Q}^{\tilde{h}^*}} \left[e^{rt - \int_0^t (\mu + i_u + \sigma_\delta h_u^P) du} dG_{t \wedge \tau_0}^{\Gamma, h^*} \right] &= \underbrace{E_t^{\mathbb{Q}^{\tilde{h}^*}} \left[(1 - F'(m_t)) d\hat{L}_t \right]}_{\leq 0} + \underbrace{E_t^{\mathbb{Q}^{\tilde{h}^*}} \left[1_{\{t = \tau_k\}} \left(F \left(m_t + \frac{e_{\tau_k}}{\rho_1} - \rho_0 \right) - F(m_t) - e_{\tau_k} \right) \right]}_{\leq 0} \\ &\quad + \underbrace{E_t^{\mathbb{Q}^{\tilde{h}^*}} \left[F'(m_t) (\alpha - g(i) - \sigma_Y \kappa_T \sin \phi + (r - \lambda)m) + \frac{1}{2} F''(m_t) (\sigma_Y^2 + m_t^2 \sigma_\delta^2) dt + F(m_t) (-r + \mu + i_t - \sigma_\delta \kappa_P \cos \phi) \right]}_{\leq 0} dt \leq 0, \end{aligned}$$

where the first inequality holds from the concavity of F and the fact that $F'(m^*) = 1$, the second from Lemma 4, and the third from the IHJB (25) and the choice of h^* . Taking expectations and letting t to ∞ yields:

$$\begin{aligned} E_0^{\mathbb{Q}^{\tilde{h}^*}} \left[G_{\tau_0}^{\Gamma, h^*} \right] &= E_0^{\mathbb{Q}^{\tilde{h}^*}} \left[\int_0^{\tau_0} e^{-rs + \int_0^s (\mu + i_u + \sigma_\delta h_u^P) du} (d\hat{L}_s - d\hat{E}_s) + e^{-r\tau_0 + \int_0^{\tau_0} (\mu + i_u + \sigma_\delta h_u^P) du} \omega q^{FB} \right] \\ &= f(m; \tau_n, e_n, L, i, h^*) \leq G_{t \wedge \tau_0}^{\Gamma, h^*} = F(m_t) \end{aligned}$$

which completes the proof of the lemma. □

Lemma 8. For any belief distortion $h = (h_t^P, h_t^T)$ and the admissible policy $\Gamma^* = (\tau_n^*, e_n^*, L^*, i^*)$ such that the dynamics of m_t are implied by (19), (28), and (30), then

$$F(m) \leq f(m; \tau_n^*, e_n^*, L^*, i^*, h) \tag{60}$$

Proof. Consider the gain process $G_{t \wedge \tau_0}^{\Gamma^*, h}$ defined by

$$G_{t \wedge \tau_0}^{\Gamma^*, h} = \int_0^{t \wedge \tau_0} e^{-rs + \int_0^s (\mu + i_u + \sigma_\delta h_u^P) du} (dL_s - dE_s) + 1_{\{t < \tau_0\}} e^{-rt + \int_0^t (\mu + i_u + \sigma_\delta h_u^P) du} F(m_t) \\ + 1_{\{t \geq \tau_0\}} e^{-r\tau_0 + \int_0^{\tau_0} (\mu + i_u + \sigma_\delta h_u^P) du} \omega q^{FB}$$

We will show that $G_{t \wedge \tau_0}^{\Gamma^*, h}$ is a \mathbb{Q}^h sub-martingale. Using Ito's formula yields

$$e^{rt - \int_0^t (\mu + i_u + \sigma_\delta h_u^P) du} dG_{t \wedge \tau_0}^{\Gamma^*, h} = (dL_t - dE_t) + F'(m_t) \left(\alpha - g(i_t) + h_t^T + (r - \lambda)m \right) dt \\ + \frac{1}{2} F''(m_t) (\sigma_Y^2 + m_t^2 \sigma_\delta^2) dt + F(m_t) \left(-r + \mu + i_t + h_t^P \right) dt + F(m_t) \sqrt{\sigma_Y^2 + m_t^2 \sigma_\delta^2} dW_t^m \\ + F'(m_t) (-dL_t) + 1_{\{t = \tau_k\}} \left(F \left(m_t + \frac{e\tau_k}{\rho_1} - \rho_0 \right) - F(m_t) \right)$$

taking expectations we obtain that

$$E_t^{\mathbb{Q}^h} \left[e^{rt - \int_0^t (\mu + i_u + \sigma_\delta h_u^P) du} dG_{t \wedge \tau_0}^{\Gamma^*, h} \right] = \underbrace{E_t^{\mathbb{Q}^h} \left[(1 - F'(m_t)) d\hat{L}_t \right]}_{\geq 0} + \underbrace{E_t^{\mathbb{Q}^h} \left[1_{\{t = \tau_k\}} \left(F \left(m_t + \frac{e\tau_k}{\rho_1} - \rho_0 \right) - F(m_t) - e\tau_k \right) \right]}_{\geq 0} \\ + \underbrace{E_t^{\mathbb{Q}^h} \left[F'(m_t) \left(\alpha - g(i) + h_t^T + (r - \lambda)m \right) + \frac{1}{2} F''(m_t) (\sigma_Y^2 + m_t^2 \sigma_\delta^2) dt + F(m_t) \left(-r + \mu + i_t + h_t^P \right) \right]}_{\geq 0} dt \geq 0,$$

where the first inequality holds from the fact that m is reflected at m^* and $F'(m^*) = 1$, the second from (29), and the third from the IHJB (25) and the choice of Γ^* . Taking expectations and letting t to ∞ yields:

$$E_0^{\mathbb{Q}^h} \left[G_{t \wedge \tau_0}^{\Gamma^*, h} \right] = E_0^{\mathbb{Q}^h} \left[\int_0^{\tau_0} e^{-rs + \int_0^s (\mu + i_u + \sigma_\delta h_u^P) du} (dL_s - dE_s) + e^{-r\tau_0 + \int_0^{\tau_0} (\mu + i_u + \sigma_\delta h_u^P) du} \omega q^{FB} \right] \\ = f(m; \tau_n^*, e_n^*, L^*, i^*, h) \geq G_{t \wedge \tau_0}^{\Gamma^*, h} = F(m_t)$$

which completes the proof of the lemma.

From (59) it follows that for any admissible policy $\Gamma = (\tau_n, e_n, L, i)$

$$\min_h f(m; \tau_n, e_n, L, i, h) \leq f(m; \tau_n, e_n, L, i, h^*) \leq F(m)$$

Maximizing over $\Gamma = (\tau_n, e_n, L, i)$ we get that

$$\max_{\Gamma} \min_h f(m; \tau_n, e_n, L, i, h) \leq \max_{\Gamma} f(m; \tau_n, e_n, L, i, h^*) \leq F(m)$$

$$F^*(m) \leq F(m). \quad (61)$$

From (60) we can minimize over h

$$F(m) \leq \min_h f(m; \tau_n^*, e_n^*, L^*, i^*, h) \leq \max_{\Gamma} \min_h f(m; \tau_n, e_n, L, i, h) \quad (62)$$

$$F(m) \leq F^*(m). \tag{63}$$

Combining (63) and (61) it follows that $F(m)$ coincides with the value function for the one dimensional control problem $F^*(m)$.

□